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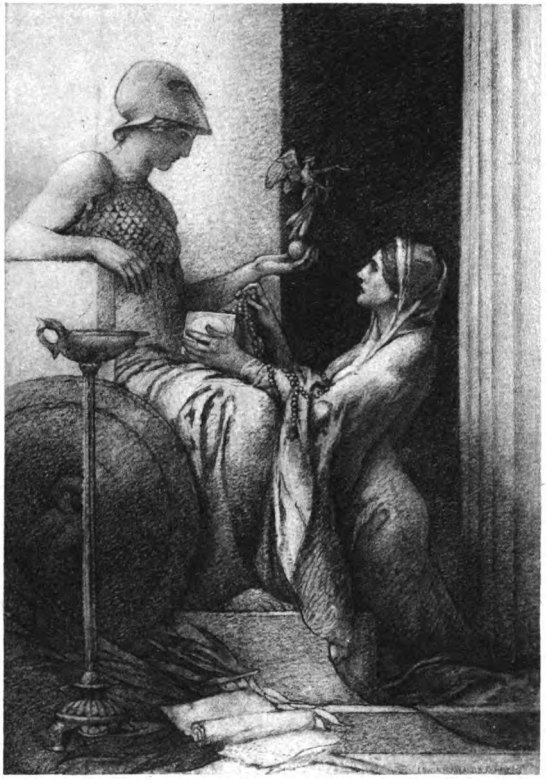

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VOLUME V.

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CAMBRIDGE PHILOSOPHICAL SOCIETY.

VOLUME V.

OCTOBER 29, 1883—MAY 24, 1886.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

ANNUAL GENERAL MEETING.

October 29, 1883.

THE following were elected officers and new members of the Council for the ensuing year:—

President:

Mr Glaisher.

Vice-Presidents:

Prof. Cayley, Prof. Stokes, Lord Rayleigh.

Treasurer:

Mr J. W. Clark.

Secretaries:

Mr Trotter, Mr Glazebrook, Mr Vines.

New Members of Council:

Prof. Humphry, Prof. Babington, Prof. Adams, Prof. Newton, Mr F. Darwin, Mr Shaw, Mr Sedgwick.

The following were ballotted for and elected Fellows of the Society:—

Mr. R. D. Roberts, M.A., Clare College,
Mr W. Warren, M.A., St John's College,
Mr W. F. B. Weldon, B.A., St John's College.

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The President in returning thanks for his re-election referred to the work of the Society during the past year. The number and importance of the communications made to the Society had rendered the year a remarkable one; and he especially noticed the papers relating to Botany, a subject which till recently had been but rarely brought before the Society. In the year a Committee of the Council had prepared a greatly extended list of Foreign Institutions to which the Transactions and Proceedings of the Society should be sent. The list had been approved by the Council, so that the publications of the Society were now widely circulated abroad, and the complaint that they were difficult to consult on the Continent could not justly be made in the future. The Council had also reprinted Vol. II., part 1, of the Transactions, so that complete sets of the Transactions could now be procured. A Committee of the Council were at present considering the question of improving and rendering more useful and available the publications of the Society, as, for example, by publishing and offering for sale all papers separately, or forming separate parts consisting only of papers relating to the same group of subjects. He regretted that the Society had lost two of their officers, Dr Pearson, and Mr Hicks, who had gone out of residence: but he was glad that Mr J. W. Clark had consented, in spite of his numerous engagements, to accept the office of Treasurer, and that the Society had obtained the services of Mr Glazebrook and Mr Vines as secretaries.

The following papers were communicated to the Society:

1. *On the effect of viscosity on the tides.* By the Rev. O. FISHER, M.A.

2. *Note on Mr Larmor's communication on "Critical Equilibrium."* By A. G. GREENHILL, M.A.

Mr Larmor has expressed the results of the integration of his differential equations in Legendre's notation, but a slight modification will exhibit them in Jacobi's notation for the direct elliptic functions.

Thus for the differential equation

$$\frac{k^2}{g}\frac{d^2\theta}{dt^2} = c\theta - m\theta^3,$$

integrating, supposing β the amplitude of vibration

$$\begin{aligned}\tfrac{1}{2}\frac{k^2}{g}\left(\frac{d\theta}{dt}\right)^2 &= -\tfrac{1}{2}c(\beta^2-\theta^2)+\tfrac{1}{4}m(\beta^4-\theta^4)\\ &= (\beta^2-\theta^2)(\tfrac{1}{4}m\beta^2-\tfrac{1}{2}c+\tfrac{1}{4}m\theta^2);\end{aligned}$$

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and therefore, supposing c small, so that $\frac{1}{4}m\beta^2 - \frac{1}{2}c$ is positive,

$$\theta = \beta \operatorname{cn} K\frac{t}{T},$$

where
$$\frac{k'^2}{k^2} = 1 - \frac{2c}{m\beta^2};$$

and therefore
$$k'^2 = \frac{1 - \frac{2c}{m\beta^2}}{2 - \frac{2c}{m\beta^2}}, \quad k^2 = \frac{1}{2 - \frac{2c}{m\beta^2}};$$

and
$$\frac{K^2k'^2}{T^2} = (\tfrac{1}{2}m\beta^2 - c)\frac{g}{k^2}$$

or
$$\frac{K^2}{T^2} = (m\beta^2 - c)\frac{g}{k^2};$$

and then $2T$ is the time of a vibration.

If $c = 0$, then
$$\frac{k'^2}{k^2} = 1, \quad k'^2 = k^2 = \tfrac{1}{2} = \sin^2 45^\circ,$$

and
$$\frac{k^2}{g}\left(\frac{d\theta}{dt}\right)^2 = \tfrac{1}{2}m(\beta^4 - \theta^4)$$

$$\theta = \beta \operatorname{cn} K\frac{t}{T},$$

where
$$\frac{k^2}{T^2} = \frac{mg\beta^2}{k^2}$$

$$T = \frac{kK}{\beta\sqrt{(mg)}},$$

where
$$K = 1{\cdot}85407.$$

Again, in the solution of

$$\frac{d^2\theta}{dt^2} = -\mu\theta^5,$$

integrating
$$\left(\frac{d\theta}{dt}\right)^2 = \tfrac{1}{3}\mu(\beta^6 - \theta^6),$$

and
$$\theta^2 = \frac{\beta^2}{1 + y^2}$$

$$= \beta^2\frac{1 + \operatorname{cn} K\frac{t}{T}}{(\sqrt{3} + 1) - (\sqrt{3} - 1)\operatorname{cn} K\frac{t}{T}}.$$

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3. *Complex Multiplication of Elliptic Functions.* By A. G. GREENHILL, M.A.

In a previous communication to the Cambridge Philosophical Society on Nov. 27, 1882, I had the honour of shewing that it was possible to express

$$y = \mathrm{cn}\,\tfrac{1}{2}\left(1 + i\frac{K'}{K}\right)u$$

in terms of $x = \mathrm{cn}\,u$,

by a rational transformation of the p^{th} order,

where $$\frac{K'}{K} = \sqrt{n},$$

when n is of the form $4p-1$, or

$$n \equiv 3 \pmod 4.$$

Denoting $\frac{k'}{k}$ by c and $\frac{K - iK'}{p}$ by ω, the transformation is of the form

$$\frac{1-y}{1+\frac{y}{ic}} = -\sqrt{(-ic)}\,\frac{1-x}{1+x}\prod_{s=1}^{s=\frac{p-1}{2}}\left\{\frac{1-\frac{x}{\mathrm{cn}\,2s\omega}}{1+\frac{x}{\mathrm{cn}\,2s\omega}}\right\}^2,$$

or $$\frac{1+y}{1-\frac{y}{ic}} = \sqrt{(-ic)}\,\frac{1-\frac{x}{ic}}{1+\frac{x}{ic}}\prod_{s=1}^{s=\frac{p-1}{2}}\left\{\frac{1-\frac{x}{\mathrm{cn}\,(2s-1)\omega}}{1+\frac{x}{\mathrm{cn}\,(2s-1)\omega}}\right\}^2,$$

if p is odd: and of the form

$$\frac{1-y}{1+\frac{y}{ic}} = -\sqrt{(-ic)}\,\frac{1-x}{1+x}\,\frac{1-\frac{x}{ic}}{1+\frac{x}{ic}}\prod_{s=1}^{s=\frac{p-2}{2}}\left\{\frac{1-\frac{x}{\mathrm{cn}\,2s\omega}}{1+\frac{x}{\mathrm{cn}\,2s\omega}}\right\}^2,$$

or $$\frac{1+y}{1-\frac{y}{ic}} = \sqrt{(-ic)}\prod_{s=1}^{s=\frac{p}{2}}\left\{\frac{1-\frac{x}{\mathrm{cn}\,(2s-1)\omega}}{1+\frac{x}{\mathrm{cn}\,(2s-1)\omega}}\right\}^2,$$

if p is even.

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These transformations shew that it is not possible to express $\mathrm{sn}\frac{1}{2}(1+i\sqrt{n})u$ in terms of $\mathrm{sn}u$, or $\mathrm{dn}\frac{1}{2}(1+i\sqrt{n})u$ in terms of $\mathrm{dn}u$, by a rational transformation, when $n \equiv 3 \pmod 4$.

For the remaining series of odd values of n, namely where n is of the form $4p+1$, or

$$n \equiv 1 \pmod 4,$$

it is not possible to express $\mathrm{cn}\frac{1}{2}(1+i\sqrt{n})u$ rationally in terms of $\mathrm{cn}u$, although it is possible to express $\mathrm{cn}(1+i\sqrt{n})u$ rationally in terms of $\mathrm{cn}u$ by a transformation of the $n+1^{\text{th}}$ order.

I have however received a letter from Mr G. H. Stuart, formerly Fellow of Emmanuel College, in which he points out that when $n=5$, that is, when

$$\frac{K'}{K} = \sqrt{5},$$

then, if $\omega = \frac{1}{3}(K - iK')$,

$$\mathrm{cn}\tfrac{1}{2}(1+i\sqrt{5})u = \frac{1-i}{\sqrt{2}}\sqrt{\frac{k'}{k}}\sqrt{\left(\frac{1-\dfrac{\mathrm{cn}\,u}{\mathrm{cn}\,3\omega}}{1+\dfrac{\mathrm{cn}\,u}{\mathrm{cn}\,3\omega}}\right)\frac{1-\dfrac{\mathrm{cn}\,u}{\mathrm{cn}\,\omega}}{1+\dfrac{\mathrm{cn}\,u}{\mathrm{cn}\,\omega}}},$$

a transformation, so to speak, of the order $1+\frac{1}{2}$.

This theorem can be immediately generalized; for if

$$x = \mathrm{cn}\,u, \quad y = \mathrm{cn}\tfrac{1}{2}(1+i\sqrt{n})u;$$

where

$$\frac{K'}{K} = \sqrt{n},$$

and n is of the form $4p+1$, or

$$n \equiv 1 \pmod 4;$$

and, if

$$c = \frac{k'}{k},$$

and

$$\omega = \frac{K - iK'}{2p+1},$$

then

$$y = \sqrt{(-ic)}\sqrt{\left(\frac{1-\dfrac{x}{ic}}{1+\dfrac{x}{ic}}\right)}\prod_{s=1}^{s=p}\frac{1-\dfrac{x}{\mathrm{cn}\,(2s-1)\,\omega}}{1+\dfrac{x}{\mathrm{cn}\,(2s-1)\,\omega}},$$

a transformation of the order $p+\frac{1}{2}$, equivalent to

$$1-y^2 = (1+ic)(1-x)\prod\left(1-\frac{x}{\mathrm{cn}\,2s\omega}\right)^2 \div D,$$

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or $$1+\frac{y^2}{c^2}=\left(1-\frac{i}{c}\right)(1+x)\prod\left(1+\frac{x}{\operatorname{cn} 2s\omega}\right)^2 \div D,$$

where $$D=\left(1+\frac{x}{ic}\right)\prod\left\{1+\frac{x}{\operatorname{cn}(2s-1)\,\omega}\right\}^2;$$

leading to the differential relation

$$\frac{dy}{\sqrt{\left(1-y^2.1+\frac{y^2}{c^2}\right)}}=\frac{\frac{1}{2}(1+i\sqrt{n})\,dx}{\sqrt{\left(1-x^2.1+\frac{x^2}{c^2}\right)}}.$$

For instance, when $n=1$, then $p=0$, $c=1$, and

$$\operatorname{cn}\tfrac{1}{2}(1+i)\,u=\sqrt{(-i)}\sqrt{\left(\frac{1-\frac{\operatorname{cn} u}{i}}{1+\frac{\operatorname{cn} u}{i}}\right)}.$$

This can easily be verified; for if we put

$$\tfrac{1}{2}(1+i)\,u=v,$$

then $$u=(1-i)\,v,$$

and then the above relation gives

$$\operatorname{cn} u=i\,\frac{1-i\operatorname{cn}^2\frac{1}{2}(1+i)\,u}{1+i\operatorname{cn}^2\frac{1}{2}(1+i)\,u},$$

or $$\operatorname{cn}(1-i)\,v=i\,\frac{1-i\operatorname{cn}^2 v}{1+i\operatorname{cn}^2 v},$$

the well-known relation, leading to

$$\frac{dy}{\sqrt{(1-y^4)}}=\frac{(1-i)\,dx}{\sqrt{(1-x^4)}},$$

if $$x=\operatorname{cn} v,\quad y=\operatorname{cn}(1-i)\,v.$$

Again, when $$\frac{K'}{K}=\sqrt{5},$$

$$2kk'=\sqrt{5}-2,$$

$$c=\sqrt{5}+2+2\sqrt{(\sqrt{5}+2)},$$

$$\sqrt{c}=\frac{\sqrt{5}+1}{2}+\sqrt{\left(\frac{\sqrt{5}+1}{2}\right)};$$

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and we have to verify that, if

$$x = \operatorname{cn} u, \quad y = \operatorname{cn} \tfrac{1}{2}(1 + i\sqrt{5})\, u,$$

$$y = \sqrt{(-ic)} \sqrt{\left(\frac{1 - \frac{x}{ic}}{1 + \frac{x}{ic}}\right)} \frac{1 - \frac{x}{\alpha}}{1 + \frac{x}{\alpha}},$$

where $\alpha = \operatorname{cn} \frac{1}{3}(K - iK')$; also

$$1 - y^2 = (1 + ic)(1 - x)\left(1 - \frac{x}{\beta}\right)^2 \div D,$$

$$1 + \frac{y^2}{c^2} = \left(1 - \frac{i}{c}\right)(1 + x)\left(1 + \frac{x}{\beta}\right)^2 \div D,$$

$$D = \left(1 + \frac{x}{ic}\right)\left(1 + \frac{x}{\alpha}\right)^2,$$

where $\beta = \operatorname{cn} \frac{2}{3}(K - iK')$,

and therefore $\alpha\beta = ic$.

Since $x = 1$ when $y = 1$, therefore

$$\frac{\alpha + 1}{\alpha - 1} = \sqrt{(-ic)} \sqrt{\left(\frac{ic - 1}{ic + 1}\right)},$$

determining α or $\operatorname{cn} \frac{1}{3}(K - iK')$.

Again $x = \beta = \frac{ic}{\alpha}$,

where $y^2 = 1$;

therefore $$1 = -ic \cdot \frac{\alpha - 1}{\alpha + 1} \cdot \left(\frac{\alpha^2 - ic}{\alpha^2 + ic}\right)^2,$$

which should be satisfied by the preceding value of α.

Differentiating y logarithmically

$$\frac{1}{y}\frac{dy}{dx} = -\frac{\frac{1}{ic}}{1 + \frac{x^2}{c^2}} - \frac{\frac{2}{\alpha}}{1 - \frac{x^2}{\alpha^2}}$$

$$= -\frac{\frac{2}{\alpha} + \frac{1}{ic} + \left(\frac{2}{\alpha c^2} - \frac{1}{ix^2 c}\right) x^2}{\left(1 + \frac{x^2}{c^2}\right)\left(1 - \frac{x^2}{\alpha^2}\right)};$$

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therefore
$$\frac{dy}{\sqrt{\left(1-y^2.1+\frac{y^2}{c^2}\right)}}$$

$$=-\frac{\left\{\frac{2}{\alpha}+\frac{1}{ic}+\left(\frac{2}{\alpha c^2}-\frac{1}{i\alpha^2 c}\right)x^2\right\}\sqrt{-ic}\,dx}{\sqrt{\left(2+ic-\frac{i}{c}\right)\left(1-\frac{x^2}{\beta^2}\right)}\sqrt{\left(1-x^2.1+\frac{x^2}{c^2}\right)}}$$

$$=-\frac{M dx}{\sqrt{\left(1-x^2.1+\frac{x^2}{c^2}\right)}},$$

where
$$M=\sqrt{-ic}\left(\frac{2}{\alpha}+\frac{1}{ic}\right)\div\sqrt{\left(2+ic-\frac{i}{c}\right)},$$

if
$$\beta^2=-\frac{c^2}{\alpha^2}=\frac{\frac{2}{\alpha}+\frac{1}{ic}}{\frac{1}{i\alpha^2 c}-\frac{2}{\alpha c^2}}$$

or
$$\alpha^4+2ic\alpha^3-2ic\alpha+c^2=0;$$

but this is the equation of which one root is
$$\alpha=\operatorname{cn}\tfrac{1}{3}(K-iK'):$$

for
$$\frac{1-\frac{\operatorname{cn}3u}{ic}}{1+\frac{\operatorname{cn}3u}{ic}}=\frac{1-\frac{\operatorname{cn}u}{ic}}{1+\frac{\operatorname{cn}u}{ic}}\left(\frac{\operatorname{cn}^4u+2ic\operatorname{cn}^3u-2ic\operatorname{cn}u+c^2}{\operatorname{cn}^4u-2ic\operatorname{cn}^3u+2ic\operatorname{cn}u+c^2}\right)^2.$$

With the above values of c and α, the value of M should reduce to
$$\tfrac{1}{2}(1+i\sqrt{5}).$$

When
$$\frac{K'}{K}=\sqrt{9},\quad 2kk'=(2-\sqrt{3})^2,$$

and
$$c=(2+\sqrt{3})^2+2\sqrt[4]{12}\,(2+\sqrt{3}),$$

$$\sqrt{c}=\frac{\sqrt{3}+1}{\sqrt{2}}(\sqrt{2}+\sqrt[4]{3}).$$

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Then, if $x = \operatorname{cn} u, \quad y = \operatorname{cn} \tfrac{1}{2}(1 + i\sqrt{9})\, u,$

$$y = \sqrt{(-ic)} \sqrt{\left(\frac{1 - \frac{x}{ic}}{1 + \frac{x}{ic}}\right)} \cdot \frac{\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)}{\left(1 + \frac{x}{\alpha}\right)\left(1 + \frac{x}{\beta}\right)},$$

where $\alpha = \operatorname{cn} \tfrac{1}{5}(K - K'), \quad \beta = \operatorname{cn} \tfrac{3}{5}(K - iK');$

and $$1 - y^2 = (1 + ic)(1 - x)\left(1 - \frac{x}{\gamma}\right)^2 \left(1 - \frac{x}{\delta}\right)^2 \div D,$$

$$1 - \frac{y^2}{c^2} = \left(1 - \frac{i}{c}\right)(1 + x)\left(1 + \frac{x}{\gamma}\right)^2 \left(1 + \frac{x}{\delta}\right)^2 \div D,$$

$$D = \left(1 + \frac{x}{ic}\right)\left(1 + \frac{x}{\alpha}\right)^2 \left(1 + \frac{x}{\beta}\right)^2 :$$

and $\gamma = \operatorname{cn} \tfrac{2}{5}(K - iK'), \quad \delta = \operatorname{cn} \tfrac{4}{5}(K - iK');$

so that $$\alpha\delta = \beta\gamma = ic.$$

If $\dfrac{K'}{K} = \sqrt{13}, \quad 2kk' = 5\sqrt{13} - 18 = \left(\dfrac{\sqrt{13} - 3}{2}\right)^3;$

and if $x = \operatorname{cn} u, \quad y = \operatorname{cn} \tfrac{1}{2}(1 + i\sqrt{13})\, u,$

$$y = \sqrt{(-ic)} \sqrt{\left(\frac{1 - \frac{x}{ic}}{1 + \frac{x}{ic}}\right)} \cdot \frac{\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)\left(1 - \frac{x}{\gamma}\right)}{\left(1 + \frac{x}{\alpha}\right)\left(1 + \frac{x}{\beta}\right)\left(1 + \frac{x}{\gamma}\right)},$$

where $\alpha = \operatorname{cn} \tfrac{1}{7}(K - iK'), \quad \beta = \operatorname{cn} \tfrac{3}{7}(K - iK'), \quad \gamma = \operatorname{cn} \tfrac{5}{7}(K - iK');$ and so on.

The superior regularity of the periods of the cn function, and of the positions of the critical points of the corresponding integral

$$\int \frac{dx}{\sqrt{\left(1 - x^2 \cdot 1 + \frac{x^2}{c^2}\right)}},$$

render the cn function preferable to the sn and dn function for employment in the general formulæ of transformation and of complex multiplication.

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For instance in the general transformation of an odd order, if

$$x = \operatorname{cn} u, \quad y = \operatorname{cn}\left(\frac{u}{M}, \ \lambda\right)$$

$$1 - y = (1 - x)\left[1 - \frac{x}{\operatorname{cn} 2s\omega}\right]^2 \div,$$

$$1 + y = (1 + x)\left[1 + \frac{x}{\operatorname{cn} 2s\omega}\right]^2 \div,$$

$$1 - \frac{y}{ic} = \left(1 - \frac{x}{ic}\right)\left[1 - \frac{y}{\operatorname{cn}(2s-1)\,\omega}\right]^2 \div,$$

$$1 + \frac{y}{ic} = \left(1 + \frac{x}{ic}\right)\left[1 + \frac{x}{\operatorname{cn}(2s-1)\,\omega}\right]^2 \div.$$

When n is an even number however, the sn function must be employed; for instance, if

$$\frac{K'}{K} = \sqrt{2}, \quad \text{then } k = \sqrt{2} - 1;$$

and if $$x = \operatorname{sn} u, \quad y = \operatorname{sn}(1 + i\sqrt{2})\, u;$$

$$y = (1 + i\sqrt{2})\, x \frac{1 - \dfrac{x^2}{\operatorname{sn}^2 2\omega}}{1 - k^2 x^2 \operatorname{sn}^2 2\omega},$$

where $$\omega = \tfrac{1}{3}(K - iK');$$

leading to the equations

$$\frac{1-y}{1+y} = \frac{1+kx}{1-kx}\left(\frac{1 - \dfrac{x}{\operatorname{sn}\omega}}{1 + \dfrac{x}{\operatorname{sn}\omega}}\right)^2,$$

$$\frac{1-ky}{1+ky} = \frac{1-x}{1+x}\left(\frac{1+kx\operatorname{sn}\omega}{1-kx\operatorname{sn}\omega}\right)^2.$$

Also $$\operatorname{sn}\omega = \sqrt{-i},$$

so that $$\operatorname{sn}^2 2\omega = \frac{\sqrt{2} - i}{\sqrt{2} - 1},$$

and $$y = (1 + i\sqrt{2})\, x \frac{1 - \dfrac{\sqrt{2}-1}{\sqrt{2}-i} x^2}{1 - (\sqrt{2}-1)(\sqrt{2}-i)\, x^2},$$

$$= \frac{(1 + i\sqrt{2})\, x - (\sqrt{2} - 1)\, ix^3}{1 - (\sqrt{2} - 1)\,(\sqrt{2} - i)\, x^2}.$$

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These transformations shew that it is not possible to express $\operatorname{cn}(1+i\sqrt{2})u$ in terms of $\operatorname{cn} u$, or $\operatorname{dn}(1+i\sqrt{2})u$ in terms of $\operatorname{dn} u$, by a rational transformation.

Again, if
$$\frac{K'}{K} = 2,$$

then
$$k = 3 - 2\sqrt{2};$$

and if
$$x = \operatorname{sn} u, \quad y = \operatorname{sn}(1+2i)u,$$

then
$$y = (1+2i)\,x\,\frac{\left(1-\frac{x^2}{\operatorname{sn}^2 2\omega}\right)\left(1-\frac{x^2}{\operatorname{sn}^2 4\omega}\right)}{(1-k^2x^2\operatorname{sn}^2 2\omega)\,(1-k^2x^2\operatorname{sn}^2 4\omega)},$$

where
$$\omega = \tfrac{1}{3}(K - iK');$$

leading to the equations

$$\frac{1-y}{1+y} = \frac{1-kx}{1+kx}\left(\frac{1-\frac{x}{\operatorname{sn}\omega}}{1+\frac{x}{\operatorname{sn}\omega}}\right)^2\left(\frac{1+\frac{x}{\operatorname{sn} 3\omega}}{1-\frac{x}{\operatorname{sn} 3\omega}}\right)^2,$$

$$\frac{1-ky}{1+ky} = \frac{1-x}{1+x}\left(\frac{1-\frac{x}{\operatorname{sn} 2\omega}}{1+\frac{x}{\operatorname{sn} 2\omega}}\right)^2\left(\frac{1+\frac{x}{\operatorname{sn} 4\omega}}{1-\frac{x}{\operatorname{sn} 4\omega}}\right)^2;$$

so that $\operatorname{cn}(1+2i)u$ has a factor $\operatorname{dn} u$, and $\operatorname{dn}(1+2i)u$ a factor $\operatorname{cn} u$.

Also, if
$$\frac{K'}{K} = \sqrt{6},$$

then
$$k = (\sqrt{3}-\sqrt{2})(2-\sqrt{3});$$

and if
$$x = \operatorname{sn} u, \quad y = \operatorname{sn}(1+i\sqrt{6})u,$$

then

$$y = (1+i\sqrt{6})\,x\,\frac{\left(1-\frac{x^2}{\operatorname{sn}^2 2\omega}\right)\left(1-\frac{x^2}{\operatorname{sn}^2 4\omega}\right)\left(1-\frac{x^2}{\operatorname{sn}^2 6\omega}\right)}{(1-k^2x^2\operatorname{sn}^2 2\omega)\,(1-k^2x^2\operatorname{sn}^2 4\omega)\,(1-k^2x^2\operatorname{sn}^2 6\omega)},$$

where
$$\omega = \tfrac{1}{7}(K - iK');$$

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leading to the equations

$$\frac{1-y}{1+y}=\frac{1+kx}{1-kx}\left(\frac{1-\frac{x}{\operatorname{sn}\omega}}{1+\frac{x}{\operatorname{sn}\omega}}\right)^2\left(\frac{1+\frac{x}{\operatorname{sn}3\omega}}{1-\frac{x}{\operatorname{sn}3\omega}}\right)^2\left(\frac{1-\frac{x}{\operatorname{sn}5\omega}}{1+\frac{x}{\operatorname{sn}5\omega}}\right)^2,$$

$$\frac{1-ky}{1+ky}=\frac{1-x}{1+x}\left(\frac{1-\frac{x}{\operatorname{sn}2\omega}}{1+\frac{x}{\operatorname{sn}2\omega}}\right)^2\left(\frac{1+\frac{x}{\operatorname{sn}4\omega}}{1-\frac{x}{\operatorname{sn}4\omega}}\right)^2\left(\frac{1-\frac{x}{\operatorname{sn}6\omega}}{1+\frac{x}{\operatorname{sn}6\omega}}\right)^2;$$

and so on.

If n is a composite number rr', then the modular equation of the nth order is obtained by combining the modular equations of the orders r and r', and the modulus when $\frac{K'}{K}=\sqrt{n}$ is obtained by putting $\lambda=k'$ is the modular equation.

But another root of the same modular equation will be the modulus when

$$\frac{K'}{K}=\sqrt{\frac{r'}{r}},$$

and by properly choosing r and r' it will be possible to represent any assigned value of $\frac{K'}{K}$.

The general problem of the "Complex Multiplication of Elliptic Functions" is then to express the sn, cn, or dn of $(aK+biK')v$ in terms of the sn, cn, or dn of Kv, where $\frac{K'}{K}=\sqrt{\frac{r'}{r}}$; but it is necessary that b should contain a factor r, so that the complex multiplier always reduces to the form

$$a+bi\sqrt{rr'}.$$

For in expressing the elliptic functions of $i\frac{K'}{K}u$ in terms of the elliptic functions of u, the *first* transformation of the order r from the modulus k to a smaller modulus λ, and then the *second* transformation of the order r' from λ to a larger modulus k' must be employed; so that if N, N' are the corresponding multipliers of the transformations,

$$N=\frac{K}{r\Lambda},\quad N'=\frac{K'}{\Lambda};$$

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and the resultant multiplier

$$M = NN' = \frac{K}{rK'};$$

so that if $u = Kv$, then $\frac{u}{M} = rK'v$.

For instance, if $\frac{k'}{k} = \sqrt{15} = \sqrt{5 \cdot 3}$,

then $\sqrt[4]{(kk')} = \sin 18°$;

but if $\frac{K'}{K} = \sqrt{\frac{5}{3}}$,

then $\sqrt[4]{(kk')} = \sin 54°$,

(Joubert, *Comptes Rendus*, t. 50);

and $2kk' = \frac{7 + 3\sqrt{5}}{16}$,

$$c = \frac{k'}{k} = (7 + 4\sqrt{3})(4 - \sqrt{15}),$$

$$\sqrt{2c} = (2 + \sqrt{3})(\sqrt{5} - \sqrt{3}).$$

Then, if $x = \operatorname{cn} Kv$, $y = \operatorname{cn} \frac{1}{2}(K + 3iK')v$,

$$\frac{1-y}{1-\frac{y}{ic}} = \sqrt{(ic)}\,\frac{1-x}{1+x} \cdot \frac{1+\frac{x}{ic}}{1-\frac{x}{ic}} \left(\frac{1+\frac{x}{\sqrt{-ic}}}{1-\frac{x}{\sqrt{-ic}}} \right)^2$$

or $$\frac{1+y}{1+\frac{y}{ic}} = \sqrt{(ic)} \left(\frac{1+\frac{x}{\alpha}}{1-\frac{x}{\alpha}} \right)^2 \left(\frac{1+\frac{x}{\beta}}{1-\frac{x}{\beta}} \right)^2,$$

where $$\alpha\beta = -ic$$

are consistent equations; and lead to

$$\frac{dy^2}{(1-y^2)\left(1+\frac{y^2}{c^2}\right)} = \frac{M^2 dx^2}{(1-x^2)\left(1+\frac{x^2}{c^2}\right)},$$

where $M = \frac{1}{2}(1 + i\sqrt{15})$.

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This has been verified arithmetically by Mr Pilkington, and his verification is here appended.

If $$c = (7+4\sqrt{3})(4-\sqrt{15}),$$

then $$\sqrt{2c} = (2+\sqrt{3})(\sqrt{5}-\sqrt{3});$$

and if

$$\frac{1-y}{y-ic} = -\frac{1}{\sqrt{ic}}\cdot\frac{1-x}{1+x}\cdot\frac{ic+x}{ic-x}\left(\frac{\sqrt{-ic}+x}{\sqrt{-ic}-x}\right)^2$$

$$= \frac{i\sqrt{i}}{\sqrt{c}}\cdot\frac{ic-x^2+(1-ic)x}{ic-x^2-(1-ic)x}\cdot\frac{ic-x^2-2x\sqrt{-ic}}{ic-x^2+2x\sqrt{-ic}}$$

$$= \frac{i-1}{\sqrt{2c}}\cdot\frac{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)+(ic-x^2)(1-\sqrt{-ie})^2x}{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)-(ic-x^2)(1-\sqrt{-ic})^2x},$$

then $1-y$

$$= \frac{(i-1+c+ic)\{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)+(ic-x^2)(1-\sqrt{-ic})^2x\}}{D}$$

where $$D = (i-1+\sqrt{2c})\{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)\}$$
$$+(i-1-\sqrt{2c})(ic-x^2)(1-\sqrt{-ic})^2x,$$

and $1+y =$

$$\frac{(i-1+2\sqrt{2c}-c-ic)\{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)\}+(i-1-2\sqrt{2c}-c-ic)(ic-x^2)(1-\sqrt{-ic})^2x}{D}.$$

This should be of the form

$$1+y = \frac{P(\alpha+x)^2(\beta+x)^2}{D}$$

$$= \frac{P\{\alpha^2\beta^2+2\alpha\beta(\alpha+\beta)x+[(\alpha+\beta)^2+2\alpha\beta]x^2+2(\alpha+\beta)x^3+x^4\}}{D},$$

whence, by comparing, we must have the following relations

$$P = i-1+2\sqrt{2c}-c-ic \quad \ldots\ldots(1),$$

$$\alpha^2\beta^2 = i^2c^2 \quad \ldots\ldots(2),$$

$$(\alpha+\beta)^2+2\alpha\beta = -2ic-2\sqrt{-ic}(1-ic) \quad \ldots\ldots(3),$$

$$2\alpha\beta(\alpha+\beta) = \frac{i-1-2\sqrt{2c}-c-ic}{i-1+2\sqrt{2c}-c-ic}\cdot ic(1-\sqrt{-ic})^2 \quad \ldots\ldots(4),$$

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$$2(\alpha+\beta) = -\frac{i-1-2\sqrt{2c}-c-ic}{i-1+2\sqrt{2c}-c-ic}(1-\sqrt{-ic})^2 \quad \ldots\ldots\ldots(5).$$

From (4) and (5) we have $\alpha\beta = -ic$, which satisfies (2); then by (3) and (5)

$$(\alpha+\beta)^2 = -2\sqrt{-ic}\,(1-ic),$$

$$2(\alpha+\beta) = -\frac{i-1-2\sqrt{2c}-c-ic}{i-1+2\sqrt{2c}-c-ic}(1-\sqrt{-ic})^2;$$

therefore we must have

$$\left(\frac{i-1-2\sqrt{2c}-c-ic}{i-1+2\sqrt{2c}-c-ic}\right)^2(1-ic-2\sqrt{-ic})^2 = -8\sqrt{-ic}\,(1-ic),$$

or $$(i-1-2\sqrt{2c}-c-ic)^2[1-ic-(1-i)\sqrt{2c}]^2$$
$$= -4(1-i)\sqrt{2c}\,(1-ic)(i-1+2\sqrt{2c}-c-ic)^2.$$

The right-hand side of this equation

$$= -2i+8\sqrt{2c}+32ic-40c\sqrt{2c}-68ic^2$$
$$+40c^2\sqrt{2c}+32ic^3-8c^3\sqrt{2c}-2ic^4,$$

and the left-hand side

$$= 8(i+1)\sqrt{2c}-64ic+40(i-1)c\sqrt{2c}$$
$$+40(i+1)c^2\sqrt{2c}-64ic^3+8(i-1)c^3\sqrt{2c},$$

therefore equating we get

$$2i+8i\sqrt{2c}-96ic+40ic\sqrt{2c}+68ic^2$$
$$+40ic^2\sqrt{2c}-96ic^3+8ic^3\sqrt{2c}+2ic^4=0,$$

or $$1+4\sqrt{2c}-48c+20c\sqrt{2c}+34c^2$$
$$+20c^2\sqrt{2c}-48c^3+4c^3\sqrt{2c}+c^4=0.$$

Substituting the above values of c and $\sqrt{2c}$, the left-hand side of this equation becomes

$$= 1-12-8\sqrt{3}+8\sqrt{5}+4\sqrt{15}-1344-768\sqrt{3}$$
$$+576\sqrt{5}+336\sqrt{15}$$
$$+c\{-60-40\sqrt{3}+40\sqrt{5}+20\sqrt{15}+952+544\sqrt{3}$$
$$-408\sqrt{5}-238\sqrt{15}\}$$

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$$+c^3\{-60-40\sqrt{3}+40\sqrt{5}+20\sqrt{15}-1344-768\sqrt{3}$$
$$+576\sqrt{5}+336\sqrt{15}\}$$
$$+c^5\{-12-8\sqrt{3}+8\sqrt{5}+4\sqrt{15}+28+16\sqrt{3}-12\sqrt{5}-7\sqrt{15}\}$$
$$=-1355-766\sqrt{3}+584\sqrt{5}+340\sqrt{15}$$
$$+c\{892+504\sqrt{3}-368\sqrt{5}-218\sqrt{15}\}$$
$$-c^2\{1404+808\sqrt{3}-616\sqrt{5}-356\sqrt{15}\}$$
$$+c^3\{16+8\sqrt{3}-4\sqrt{5}-3\sqrt{15}\}$$
$$=-1355-776\sqrt{3}+584\sqrt{5}+340\sqrt{15}$$
$$+c\{892+504\sqrt{3}-368\sqrt{5}-218\sqrt{15}\}$$
$$-c^2\{17+8\sqrt{3}\}$$
$$=-1355-776\sqrt{3}+584\sqrt{5}+340\sqrt{15}\}$$
$$+c\{32+8\sqrt{3}+4\sqrt{5}-3\sqrt{15}\}$$
$$=-1355-776\sqrt{3}+584\sqrt{5}+340\sqrt{15}$$
$$+1355+776\sqrt{3}-584\sqrt{5}-340\sqrt{15}$$
$$=0.$$

Therefore the equations for α and β are consistent, and therefore $1+y$ can be put into the required form.

Thus
$$1+y=\frac{P(\alpha+x)^2(\beta+x)^2}{D},$$

where
$$P=i-1+2\sqrt{2c}-c-ic,$$
$$D=(i-1+\sqrt{2c})\{(ic-x^2)^2-2x^2\sqrt{-ic}(1-ic)\}$$
$$+(i-1-\sqrt{2c})(ic-x^2)(1-\sqrt{-ic})^2x,$$

and α and β are given by the equations

$$\alpha\beta=-ic$$
$$2(\alpha+\beta)=-\frac{i-1-2\sqrt{2c}-c-ic}{i-1+2\sqrt{2c}-c-ic}(1-\sqrt{-ic})^2.$$

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Again from the original equation, we have

$$ic - y = \frac{(ic-1)\sqrt{2c}\{(ic-x^2)^2 - 2x^2\sqrt{-ic}(1-ic) - (ic-x^2)(1-\sqrt{-ic})^2 x\}}{D},$$

therefore $ic + y =$

$$\frac{(ic\sqrt{2c} - 2c - 2ic + \sqrt{2c})\{(ic-x^2)^2 - 2x^2\sqrt{-ic}(1-ic)\} - (\sqrt{2c} + ic\sqrt{2c} + 2c + 2ic)(ic-x^2)(1-\sqrt{-ic})^2 x}{D}.$$

This will be in the form

$$ic + y = \frac{Q(\alpha - x)^2(\beta - x)^2}{D}$$

where $Q = ic\sqrt{2c} - 2c - 2ic + \sqrt{2c}$,

provided we have

$$\frac{\sqrt{2c} + ic\sqrt{2c} + 2c + 2ic}{\sqrt{2c} + ic\sqrt{2c} - 2c - 2ic} = \frac{i - 1 - 2\sqrt{2c} - c - ic}{i - 1 + 2\sqrt{2c} - c - ic}.$$

This gives

$$\frac{\sqrt{2c} + ic\sqrt{2c}}{2c + 2ic} = \frac{i - 1 - c - ic}{-2\sqrt{2c}},$$

$$\frac{1 + ic}{1 + i} = \frac{i - 1 - c - ic}{-2},$$

$$-2(1 + ic) = (1 + i)\{(i - 1) - c(i + 1)\}$$

$$= -2 - 2ic$$

an identity.

Therefore $ic + y$ takes the required form.

Thus

$$1 + y = \frac{P(\alpha + x)^2(\beta + x)^2}{D},$$

$$ic + y = \frac{Q(\alpha - x)^2(\beta - x)^2}{D};$$

therefore

$$\frac{1 + y}{ic + y} = \frac{P(\alpha + x)^2(\beta + x)^2}{Q(\alpha - x)^2(\beta - x)^2},$$

where

$$\frac{P}{Q} = \frac{i - 1 + 2\sqrt{2c} - c - ic}{ic\sqrt{2c} - 2c - 2ic + \sqrt{2c}}$$

$$= \frac{i - 1 + 2\sqrt{2c} - (1 + i)c}{\sqrt{2c}\{ic + 1 - \sqrt{2c}(1 + i)}$$

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$$= \sqrt{\frac{i}{c}} \cdot \frac{i - i\sqrt{2c}\sqrt{2i} - c}{ic + 1 - \sqrt{2c}(1+i)}$$

$$= \frac{i\sqrt{i}}{\sqrt{c}}$$

$$= -\frac{1}{\sqrt{ic}};$$

therefore $$\frac{1+y}{ic+y} = -\frac{1}{\sqrt{ic}} \frac{(\alpha+x)^2(\beta+x)^2}{(\alpha-x)^2(\beta-x)^2},$$

which proves the first part.

We have then

$$(1-y)D = -(1-ic)(i-1)(1-x)(ic+x)(\sqrt{-ic}+x)^2,$$

$$(1+y)D = P(\alpha+x)^2(\beta+x)^2,$$

$$(ic-y)D = -(ic-1)\sqrt{2c}(1+x)(ic-x)(\sqrt{-ic}-x)^2,$$

$$(ic+y)D = Q(\alpha-x)^2(\beta-x)^2;$$

therefore $$(1-y^2)(y^2+c^2)D^4$$

$$= PQ(1-i)\sqrt{2c}(1-ic)^2(1-x^2)(x^2+c^2)(\alpha^2-x^2)^2(\beta^2-x^2)^2(ic+x^2)^2,$$

$$\frac{(1-y^2)(y^2+c^2)}{(1-x^2)(x^2+c^2)} = \frac{PQ}{D^4}(1-i)\sqrt{2c}(1-ic)^2(\alpha^2-x^2)^2(\beta^2-x^2)^2(ic+x^2)^2$$

$$= -\frac{2P^2c(1-ic)^2(\alpha^2-x^2)^2(\beta^2-x^2)^2(ic+x^2)^2}{D^4} \ldots (A).$$

Again, since

$$1-y = \frac{i-1+c+ic}{D}\{(ic-x^2)^2 - 2x^2\sqrt{-ic}(1-ic) + (ic-x^2)(1-\sqrt{-ic})^2x\},$$

we have

$$\frac{dy}{dx} = \frac{i-1+c+ic}{D^2}$$

$$\times \Big\{ D[4x(ic-x^2) + 4x\sqrt{-ic}(1-ic) - (ic-3x^2)(1-\sqrt{-ic})^2]$$

$$+ [(ic-x^2)^2 - 2x^2\sqrt{-ic}(1-ic) + (ic-x^2)(1-\sqrt{-ic})^2x]\frac{dD}{dx}$$

$$= \frac{i-1+c+ic}{D^2}$$

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$$\times \left\{(ic-x^2)\left[4Dx+(ic-x^2)\frac{dD}{dx}\right]+2r\sqrt{-ic}(1-ic)\left[2D-x\frac{dD}{dx}\right]\right.$$
$$\left.-(1-\sqrt{-ic})^2\left[(ic-3x^2)D-(ic-x^2)\,x\frac{dD}{dx}\right]\right\}.$$

Now $D=(i-1+\sqrt{2c})\{(ic-x^2)^2-2x^2\sqrt{-ic}\,(1-ic)\}$
$$+(i-1-\sqrt{2c})(ic-x^2)(1-\sqrt{-ic})^2\,x,$$

$$\frac{dD}{dx}=(i-1+\sqrt{2c})\{-4(ic-x^2)\,x-4x\sqrt{-ic}\,(1-ic)\}$$
$$+(i-1-\sqrt{2c})(ic-3x^2)(1-\sqrt{-ic})^2;$$

therefore

$$4Dx+(ic-x^2)\frac{dD}{dx}=(i-1+\sqrt{2c})\sqrt{-ic}\,(1-ic)(-4x)(ic+x^2)$$
$$+(i-1-\sqrt{2c})(ic-x^2)(1-\sqrt{-ic})^2(ic+x^2),$$

$$2D-x\frac{dD}{dx}=2(i-1+\sqrt{2c})(ic-x^2)(ic+x^2)$$
$$+(i-1-\sqrt{2c})(1-\sqrt{-ic})^2\,x\,(ic+x^2),$$

$$(ic-3x^2)D-(ic-x^2)\,x\frac{dD}{dx}$$
$$=(i-1+\sqrt{2c})(ic+x^2)\{(ic-x^2)^2+2x^2\sqrt{-ic}\,(1-ic)\};$$

therefore $$\frac{dy}{dx}=\frac{i-1+c+ic}{D^2}(ic+x^2)$$
$$\times\{(ic-x^2)[(i-1+\sqrt{2c})\sqrt{-ic}\,(1-ic)\,(-4x)$$
$$+(i-1-\sqrt{2c})(ic-x^2)(1-\sqrt{-ic})^2]$$
$$+2x\sqrt{-ic}\,(1-ic)[2(i-1+\sqrt{2c})(ic-x^2)$$
$$+(i-1-\sqrt{2c})(1-\sqrt{-ic})^2\,x]$$
$$-(1-\sqrt{-ic})^2(i-1+\sqrt{2c})[(ic-x^2)^2$$
$$+2x^2\sqrt{-ic}\,(1-ic)]\}$$

$$=\frac{i-1+c+ic}{D^2}(ic+x^2)\{(ic-x^2)^2(1-\sqrt{-ic})^2(-2\sqrt{2c})$$
$$+2x^2\sqrt{-ic}\,(1-ic)(1-\sqrt{-ic})^2(-2\sqrt{2c})\}$$

$$=\frac{i-1+c+ic}{D^2}(ic+x^2)(1-\sqrt{-ic})^2(-2\sqrt{2c})\{(ic-x^2)^2$$
$$+2x^2\sqrt{-ic}\,(1-ic)\}$$

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$$= -2.\frac{i-1+c+ic}{D^2}(ic+x^2)(1-\sqrt{-ic})^2\sqrt{2c}\{(\alpha\beta+x^2)^2 - (\alpha+\beta)^2x^2\}$$

$$= -2\frac{i-1+c+ic}{D^2}(ic+x^2)(1-\sqrt{-ic})^2\sqrt{2c}(\alpha^2-x^2)(\beta^2-x^2)$$

$$= -\frac{2(i-1)(1-ic)}{D^2}(ic+x^2)(1-\sqrt{-ic})^2\sqrt{2c}(\alpha^2-x^2)(\beta^2-x^2);$$

therefore

$$\frac{(1-ic)\sqrt{2c}(\alpha^2-x^2)(\beta^2-x^2)(ic+x^2)}{D^2} = -\frac{1}{2(i-1)(1-\sqrt{-ic})^2}\cdot\frac{dy}{dx}.$$

Substituting in (A), we have

$$\frac{(1-y^2)(y^2+c^2)}{(1-x^2)(x^2+c^2)} = -\frac{P^2}{4(i-1)^2(1-\sqrt{-ic})^4}\cdot\left(\frac{dy}{dx}\right)^2,$$

or
$$\left(\frac{dy}{dx}\right)^2\frac{(1-x^2)(x^2+c^2)}{(1-y^2)(y^2+c^2)} = -\frac{4(i-1)^2(1-\sqrt{-ic})^4}{P^2}$$

$$= \left[\frac{2i(i-1)(1-ic-2\sqrt{-ic})}{i-1+2\sqrt{2c}-c-ic}\right]^2$$

$$= \left[\frac{2i(1-ic-2\sqrt{-ic})}{1-(i+1)\sqrt{2c}+ic}\right]^2$$

$$= \left[\frac{2\{c-\sqrt{2c}+i(1-\sqrt{2c})\}}{1-\sqrt{2c}+i(c-\sqrt{2c})}\right]^2$$

$$= \left[\frac{2\{31+18\sqrt{3}-14\sqrt{5}-8\sqrt{15}+i(4+2\sqrt{3}-2\sqrt{5}-\sqrt{15})\}}{4+2\sqrt{3}-2\sqrt{5}-\sqrt{15}+i(31+18\sqrt{3}-14\sqrt{5}-8\sqrt{15})}\right]^2$$

$$= \left[\frac{1+i\sqrt{15}}{2}\right]^2;$$

therefore
$$\frac{dy}{\sqrt{(1-y^2)(y^2+c^2)}} = \frac{\pm\frac{1}{2}(1+i\sqrt{15})\,.\,dx}{\sqrt{(1-x^2)(x^2+c^2)}},$$

which proves the second part.

As another example of the case when $\frac{K'}{K} = \sqrt{n}$, where n is a composite number, suppose $n = 6$.

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Then, if $$\frac{K'}{K} = \sqrt{6},$$

$$k = (\sqrt{3} - \sqrt{2})\,(2 - \sqrt{3}),$$

a case already considered.

But if $$\frac{K'}{K} = \sqrt{\frac{2}{3}},$$

$$k = (\sqrt{3} + \sqrt{2})\,(2 - \sqrt{3});$$

and if $$x = \operatorname{sn} Kv, \quad y = \operatorname{sn}(K + 3iK')\,v;$$

then

$$y = (1 + i\sqrt{6})\,x\,\frac{\left(1 - \frac{x^2}{\operatorname{sn}^2 2\omega}\right)\left(1 - \frac{x^2}{\operatorname{sn}^2 4\omega}\right)\left(1 - \frac{x^2}{\operatorname{sn}^2 6\omega}\right)}{(1 - k^2x^2\operatorname{sn}^2 2\omega)\,(1 - k^2x^2\operatorname{sn}^2 4\omega)\,(1 - k^2x^2\operatorname{sn}^2 6\omega)},$$

where $$\omega = \tfrac{1}{7}(K - 3iK').$$

In the preceding transformations the factor $\frac{1}{2}$ has been introduced in the numbers a and b of the complex multipliers $a + bi\sqrt{n}$; it is interesting to see how other factors like $\frac{1}{3}$, $\frac{1}{4}$,... may come in.

For instance, if in the expression of $y = \operatorname{cn} \frac{1}{2}(1 + i\sqrt{5})\,u$ in terms of $x = \operatorname{cn} u$, we put

$$\tfrac{1}{2}(1 + i\sqrt{5})\,u = v,$$

then $$u = \tfrac{1}{3}(1 - i\sqrt{5})\,v,$$

and the same expression will give $x = \operatorname{cn} \frac{1}{3}(1 - i\sqrt{5})\,v$ in terms of $y = \operatorname{cn} v$ by the solution of a cubic equation.

Again, if $z = \operatorname{cn} \frac{1}{3}(1 + i\sqrt{5})\,v$, then

$$y = \sqrt{(-ic)}\sqrt{\left(\frac{1 - \frac{x}{ic}}{1 + \frac{x}{ic}}\right)}\,\frac{1 - \frac{x}{\alpha}}{1 + \frac{x}{\alpha'}}, \quad \alpha = \operatorname{cn} \tfrac{1}{3}(K - iK'),$$

$$= \sqrt{(ic)}\sqrt{\left(\frac{1 + \frac{z}{ic}}{1 - \frac{z}{ic}}\right)}\,\frac{1 - \frac{z}{\alpha'}}{1 + \frac{z}{\alpha'}}, \quad \alpha' = \operatorname{cn} \tfrac{1}{3}(K + iK'),$$

connecting $$x = \operatorname{cn} \tfrac{1}{3}(1 - i\sqrt{5})\,v \text{ or } x = \operatorname{cn} u,$$

and $$z = \operatorname{cn} \tfrac{1}{3}(1 + i\sqrt{5})\,v \text{ or } z = \operatorname{cn} \tfrac{1}{3}(2 - i\sqrt{5})\,u;$$

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and equivalent to the transformation

$$\frac{1-\frac{x}{ic}}{1+\frac{x}{ic}}\left(\frac{1-\frac{x}{\alpha}}{1+\frac{x}{\alpha}}\right)^2 \cdot \frac{1-\frac{z}{ic}}{1+\frac{z}{ic}}\left(\frac{1+\frac{z}{\alpha'}}{1-\frac{z}{\alpha'}}\right)^2 = -1,$$

a transformation which is capable of affording an independent verification.

For $$\frac{1-y^2}{1+\frac{y^2}{c^2}} = \frac{1+ic}{1-\frac{i}{c}} \cdot \frac{1-x}{1+x}\left(\frac{1-\frac{x}{\beta}}{1+\frac{x}{\beta}}\right)^2, \quad \beta = \mathrm{cn}\,\tfrac{2}{3}(K-iK'),$$

$$= \frac{1-ic}{1+\frac{i}{c}} \cdot \frac{1-z}{1+z}\left(\frac{1-\frac{z}{\beta'}}{1+\frac{z}{\beta'}}\right)^2, \quad \beta' = \mathrm{cn}\,\tfrac{2}{3}(K+iK');$$

and therefore

$$\frac{1-x}{1+x}\left(\frac{1-\frac{x}{\beta}}{1+\frac{x}{\beta}}\right)^2 = -\frac{1-z}{1+z}\left(\frac{1-\frac{z}{\beta'}}{1+\frac{z}{\beta'}}\right)^2,$$

or
$$\frac{1-\mathrm{cn}\left(\frac{u}{M},\lambda\right)}{1+\mathrm{cn}\left(\frac{u}{M},\lambda\right)} = -\frac{1-\mathrm{cn}\left(\frac{w}{M'},\lambda'\right)}{1+\mathrm{cn}\left(\frac{w}{M'},\lambda'\right)},$$

or,
$$\mathrm{cn}\left(\frac{u}{M},\lambda\right)\mathrm{cn}\left(\frac{w}{M'},\lambda'\right) = 1,$$

where
$$w = \tfrac{1}{3}(2-i\sqrt{5})\,u.$$

In the same way, from the expression of $y = \mathrm{cn}\,\frac{1}{2}(1+i\sqrt{7})\,u$ in terms of $x = \mathrm{cn}\,u$, we can obtain $x = \mathrm{cn}\,\frac{1}{4}(1-i\sqrt{7})\,v$ in terms of $y = \mathrm{cn}\,v$ by the solution of a quadratic; and also from the expression of $y = \mathrm{cn}\,\frac{1}{2}(1+i\sqrt{15})\,u$ in terms of $x = \mathrm{cn}\,u$, we can obtain $x = \mathrm{cn}\,\frac{1}{8}(1-i\sqrt{15})\,v$ in terms of $y = \mathrm{cn}\,v$, also by the solution of a quadratic equation.

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(4) *On some General Equations which include the Equations of Hydrodynamics.* By M. J. M. HILL, M.A.

[*Abstract.*]

1. If $u_1 u_2 \ldots u_n \rho$ be $(n+1)$ functions of $x_1 x_2 \ldots x_n t$ which satisfy

(α) the n equations which may be obtained by changing r successively into 1, 2, 3...n in

$$\frac{du_r}{dt} + u_1 \frac{du_r}{dx_1} + u_2 \frac{du_r}{dx_2} + \ldots + u_n \frac{du_r}{dx_n} = -\frac{d}{dx_r}\left(\int \frac{dp}{\rho} + V\right),$$

and (β) the equation

$$\frac{d\rho}{dt} + \frac{d}{dx_1}(\rho u_1) + \frac{d}{dx_2}(\rho u_2) + \ldots + \frac{d}{dx_n}(\rho u_n) = 0;$$

it is proved that

$$u_1 dx_1 + u_2 dx_2 + \ldots + u_n dx_n$$

$$= dK + f_1 dP_1 + f_2 dP_2 + \ldots + f_n dP_n,$$

where K satisfies the equation

$$-\int \frac{dp}{\rho} - V + \frac{u_1^2 + u_2^2 + \ldots + u_n^2}{2}$$

$$= \left(\frac{d}{dt} + u_1 \frac{d}{dx_1} + u_2 \frac{d}{dx_2} + \ldots + u_n \frac{d}{dx_n}\right) K,$$

and where $P_1 P_2 \ldots P_n$ are n independent integrals of the equation

$$\frac{df}{dt} + u_1 \frac{df}{dx_1} + u_2 \frac{df}{dx_2} + \ldots + u_n \frac{df}{dx_n} = 0;$$

and $f_1 f_2 \ldots f_s$ are arbitrary functions of $P_1 P_2 \ldots P_n$.

2. If $\xi_{rs} = \dfrac{du_r}{dx_s} - \dfrac{du_s}{dx_r}$ and if $\xi_1 \xi_2 \ldots \xi_n$ be square roots of the coefficients of $\xi_{11}\xi_{22} \ldots \xi_{nn}$ in the determinant

$$\Xi = \begin{vmatrix} \xi_{11}\xi_{12} \cdots \xi_{1n} \\ \xi_{21}\xi_{22} \cdots \xi_{2n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \xi_{n1}\xi_{n2} \cdots \xi_{nn} \end{vmatrix}$$

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it is shewn that if n be odd

$$\frac{d\xi_1}{dx_1}+\frac{d\xi_2}{dx_2}+\ldots+\frac{d\xi_n}{dx_n}=0,$$

and n equations, of which the following is a type,

$$\left(\frac{d}{dt}+u_1\frac{d}{dx_1}+u_2\frac{d}{dx_2}+\ldots+u_n\frac{d}{dx_n}\right)\left(\frac{\xi_r}{\rho}\right)$$
$$=\frac{\xi_1}{\rho}\frac{du_r}{dx_r}+\frac{\xi_2}{\rho}\frac{du_r}{dx_2}+\ldots+\frac{\xi_n}{\rho}\frac{du_r}{dx_n}.$$

When n is even, these reduce to identities and are replaced by the single equation

$$\left(\frac{d}{dt}+u_1\frac{d}{dx_1}+u_2\frac{d}{dx_2}+\ldots+u_n\frac{d}{dx_n}\right)\left(\frac{\sqrt{\Xi}}{\rho}\right)=0.$$

3. Adopting the language of Fluid Motion it is shewn that the vortex lines

$$\frac{dx_1}{\xi_1}=\frac{dx_2}{\xi_2}=\ldots=\frac{dx_n}{\xi_n}$$

always contain the same particles.

4. It is shewn that $u_1dx_1+\ldots+u_ndx_n$ (if n be odd) may always be reduced to the form

$$dK+P_{\frac{n+1}{2}}dP_1+P_{\frac{n+3}{2}}dP_2+\ldots+P_{n-1}dP_{\frac{n+1}{2}},$$

which is Clebsch's form (see his paper "Ueber die Integration der hydrodynamischen Gleichungen." *Crelle*, Bd. LVI.); and the meaning of this form is that the vortex lines are the intersections of the $(n-1)$ loci

$$P_1, P_2, \ldots P_{\frac{n-1}{2}}, P_{\frac{n+1}{2}}\ldots P_{n-1}, = \text{constant, respectively.}$$

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November 12, 1883.

Mr Glaisher, President, in the Chair.

The following communications were made to the Society:

(1) *On the Structure of Secretory Cells and on the Changes which take place in them during Secretion.* By J. N. Langley, M.A., F.R.S.

We have, from different observers, different descriptions of the structure of each of the various kinds of gland-cells. Moreover, very different accounts are given of the changes which take place during secretion, not only in the various kinds of gland-cells, but also in gland-cells of the same kind.

And this is not unnaturally the case, since different observers have examined the gland-cells under different conditions; in some instances the cells have been examined in the fresh state, in others after treatment with osmic acid or with alcohol or with chromic acid.

But scarcely any attempt has been made to reconcile these various accounts, or to ascertain what are the common points of structure, and the common changes which take place during secretion.

This I wish to do here, but to do very briefly, since I trust soon to give a more detailed account, accompanied by figures of the different glands. For this reason also, I may perhaps be allowed to confine myself to a statement of conclusions without pointing out how far they coincide or clash with the conclusions of previous observers.

The glands of vertebrates in which I find that the secretory cells have fundamentally the same structure are; the serous and mucous salivary glands and the similar glands of the mucous membrane of the mouth, nose, pharynx, œsophagus, etc.; a few exceptions with the chief cells of mammalian gastric glands; the gastric glands of such birds, fishes, reptiles and amphibia as I have examined; the œsophageal glands of the frog; the pancreas; the liver.

It will be seen that this list includes most of the secretory glands of vertebrates. The mammary glands and sweat glands of mammals, I have not yet sufficiently investigated to be certain whether they have the same structure as the preceding. The intestinal glands and the kidney, I omit for the present, since they have some special points of structure depending upon their special function of absorption and excretion respectively.

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The border-cells (Belegzellen) of mammalian gastric glands, the pyloric gland-cells, and those of the chief cells which are not distinctly granular in life—the exceptions mentioned above—I shall consider separately.

The secretory cells of all the glands in the list which I have given, have the following common points of structure[1].

The cell substance is composed of (*a*) a framework of living substance or protoplasm, connected at the periphery with a thin continuous layer of modified protoplasm; the framework in some cases has the form of a network of small threads of equal size as described by Klein[2]; in others of flattened bands. Further the threads or bands may vary in size in different parts of the cell, and the meshes in different parts of the cell may be of different size and shape. Within the meshes of the framework are enclosed two chemical substances at least, viz. (*b*) a hyaline substance in contact with the framework, and of (*c*) spherical granules which are embedded in the hyaline substance.

In the gland-cells which secrete much organic matter the cell-granules are conspicuous and fairly large. In the gland-cells which secrete comparatively little organic matter, the cell-granules are generally speaking smaller and less distinct, the lower the mean percentage of organic matter is in the fluid secreted.

The cell-granules are in nearly all cases mesostates, i.e. substances stored up in the cell and destined to give rise to the organic substances of the secretion. The granularity of a cell in the resting state thus depends upon its storage-power. Generally speaking the greater the storage-power of a cell, the higher is the percentage of organic substance in its secretion, but this is not always the case, since it may happen that the rate of secretion of water may increase without any corresponding increase in the rate of secretion of organic substance, and in consequence the percentage of organic substance in the secretion may be small; further it is possible that under special circumstances a cell with small storage-power might secrete a large quantity of its stored-up material and that a cell with large storage-power might secrete a very small quantity of its stored-up material, the amount of water secreted by the two cells being approximately equal.

In all these cells, during active secretion, the following changes take place. The granules decrease in number and usually, if not always, in size; the hyaline substance increases in amount; the network grows. The increase of the network is much less than that of the hyaline substance.

Moreover in the majority of the cells, the details of the changes which take place are much the same. The hyaline substance

[1] This I have already described for the liver-cells (*Proc. Roy. Soc.* No. 220, Ap. 1882).

[2] *Quart. Journ. Mic. Science*, Ap. 1879.

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increases chiefly in the outer region of the cells, and the granules disappear from this region; so that an outer non-granular zone and an inner granular zone are formed. The network stretches throughout the cell in all cases: in the outer zone its meshes are filled with hyaline substance; in the inner zone its meshes are filled with granules and a small amount of hyaline substance.

The glands in which an outer non-granular zone is not formed[1] during secretion are, most of the gastric glands of the frog and toad; the gastric glands of the snake, and the liver of mammals.

In the gastric glands of the snake, the decrease in the granules, and the increase of the hyaline substance, is equal or nearly equal in all parts of the cell. In the gastric glands of the frog and toad, whilst the same changes are most marked in all parts of the cell, they go on most rapidly in a narrow strip next the lumen. In the liver cells of mammals, the changes are most active in the central part of the cells around the nucleus.

It is to be remembered that there is reason to believe, that the three parts of the cell are continually being formed and changed into other substances; the extent of the change which can be observed in a cell during secretion depends upon the relative rates at which these processes go on. I have previously pointed out[2] that different gland-cells vary considerably with regard to the different relative rates at which the formation and breaking down of their constituents take place.

The differences shown by the different cells after the same treatment, depends, partly upon the different chemical characters of the framework, hyaline substance, and granules in the different cells, and partly upon the different arrangement of these constituents. With regard to the former of these causes of difference a few instances may be given.

In the rabbit's sub-maxillary gland, after treatment with osmic acid, the granules are indistinguishably mixed with the hyaline substance; and the resulting mass differs so little in refractive and staining power from the network, that the nodal points only of the network are at all distinctly seen.

In the chief-cells of the cat's gastric glands, after treatment with osmic acid, the granules and hyaline substance are also indistinguishable; but the network is much more distinct than the network in the rabbit's sub-maxillary gland. The network is however much less distinct than in a gland that has been treated with chromic acid.

In the chief-cells of the bat's gastric glands, after treatment with osmic acid, the granules are perfectly distinct, but they are

[1] The gastric glands of birds have not as yet been examined for the changes occurring in digestion. Cf. however Nussbaum, *Arch. f. mic. Anat.* XXI. p. 297, 1882.

[2] *Trans. Royal. Soc.* Part III. 1881.

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apparently embedded in a homogeneous mass, which other reagents show to be composed of network and hyaline substance.

The same cells, after treatment with alcohol, show an indistinct network, containing an interfibrillar mass in which the granules and the hyaline substance cannot be separately seen.

The œsophageal glands of the frog and the pancreas of all animals, after treatment with alcohol, show the shrunken remains of the granules, but leave the network and hyaline substance indistinguishable or nearly so.

Even chromic acid which in most gland-cells brings out the network clearly, does not act in quite the same manner on all gland-cells, for it differentiates the network and hyaline substance less clearly in the œsophageal glands of the frog, than in the salivary and gastric cells; and differentiates them less clearly in the pancreas than in the œsophageal glands of the frog.

It will be noticed that the cells mentioned above form a series in which the network and hyaline substance are less and less easily distinguished from one another, that is, a series in which the network and hyaline substance become more and more alike in chemical characters.

In the above general description I have not included the pyloric gland-cells, the border cells (Belegzellen) of mammalian gastric glands, or the semi-transparent chief-cells which are found in the latter part of the greater curvature in some animals. In these the changes described above as taking place in digestion have not yet been observed. This, I think, is due to their containing very small granules, which are not obvious during life, and which are not preserved by any reagent; in consequence a change in their granularity is very difficult to observe.

With regard to their structure they certainly have a framework enclosing hyaline substance, the only difficulty is to show that they contain also granules embedded in the hyaline substance.

These cells in life do not show distinct granules, but when they are teased out in salt solution they become very finely granular. This is not caused by the cell network, for the network has rather large meshes. In those chief-cells which are apparently homogeneous in life, the granules are often fairly distinct on teasing out the cells in salt solution; in the pyloric gland-cells the granules are usually so indistinct that I should not feel justified on the microscopic appearances alone, in assuming that they are not due to a slight alteration in the hyaline interfibrillar substance. There are however other grounds which render, I think, this assumption justifiable.

In all cells which contain much pepsinogen, distinct granules are present. Further, the quantity of pepsinogen varies directly with the mass (number and size) of the granules; that is, pepsi-

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nogen when present in cells in sufficient quantity to be readily observed is present in the form of granules, hence it seems probable that in cells which contain a small amount only of pepsinogen, the pepsinogen is also present in the form of granules which however are not conspicuous on account of their small size.

And in fact in most cases, the less pepsinogen a cell contains the smaller are its granules; this is especially well seen in the gastric glands of lower vertebrates, in some of these moreover the glands near the pyloric region may be semi-transparent and apparently homogeneous during life whilst after treatment with osmic acid, granules become obvious, which except as regards size are like the granules, granules which are of the anterior region of the stomach visible in life in the cells. Since the granules are in these cases preserved by osmic acid, their detection is easy.

It is then probable that pepsinogen when present in a cell is present in the form of granules, and that when pepsinogen is present in small quantity the granules will be too small to be easily seen. Now the semi-transparent chief-cells of the posterior gastric glands of the rabbit do contain pepsinogen, but they contain comparatively little; and the still more transparent pyloric gland-cells also contain pepsinogen but they contain much less than the chief-cells; hence I conclude that the granules which are fairly well seen in the one, and indistinctly in the other, in fresh teased out specimens, are really pepsinogen granules comparable to those easily seen in the majority of the chief-cells of mammalian gastric glands.

With regard to the border cells, there is no satisfactory proof that they contain pepsinogen, and the evidence for the presence of granules in these rests simply on the granular appearance of the fresh teased out cells, evidence which I readily admit to be anything but conclusive.

I may now pass to consider how the statement given above of the changes which take place in the cells during digestion harmonizes with the description given by Heidenhain[1] and others.

The serous cells, the mucous cells and the chief-cells of mammalian gastric glands, after treatment with alcohol, are described as being more granular and as staining better, in the active than in the resting state; that is, during secretion there is an increase of granular substance staining with carmine, and a decrease of substance not staining with carmine. The granular staining substance, Heidenhain considers to be protoplasmic substance; with this I agree, except that I consider the apparent granules of alcohol specimens to be parts of the cell network indistinctly seen, so that I take the increase of staining substance in the cells

[1] Cf. *Handbuch. d. Physiol.* (Hermann), Bd. v. 1880.

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to be the expression of a growth and rearrangement of the cell network.

The non-staining substance is considered by Heidenhain, to be substance stored up for secretory purposes, and comparable to the zymogen granules of the pancreas; with which I agree in part only; I consider the non-staining substance to consist of hyaline and of granular interfibrillar substance, the latter only corresponding to the zymogen granules of the pancreas. In all these cases, as in the pancreas, the granules disappear from the outer parts of the cells during secretion, but in alcohol specimens this cannot be observed. The active pancreatic cells differ in appearance in stained alcohol specimens from the serous and other cells mentioned above, chiefly because, in the pancreatic cells, the hyaline interfibrillar substance as well as the network takes up the colouring matter.

Another change has been described by Grützner[1] and by myself[1] as occurring in various cells during secretion. The cells, after they have been actively secreting, take a darker and browner tint on treatment with osmic acid, than they take on similar treatment after a period of rest. Formerly I referred this to the whole of the non-granular part of the cell, in which I did not then distinguish a network and hyaline substance. The change of staining power shown by the cells during secretion is however, I think, due to a change in the hyaline substance, and not to a change in the network. It is chiefly caused by the increased amount of hyaline substance. I say chiefly, since it may be partly due to the fluid, which permeates the cell, containing during secretion a greater proportion of substance capable of reducing osmic acid than it contains during rest.

The question now naturally occurs, What is the nature of the hyaline interfibrillar substance? We have seen that as the granules diminish, the hyaline substance increases, and that as the granules increase, the hyaline substance diminishes; so that an obvious hypothesis is that the protoplasmic network forms the hyaline substance and then out of this manufactures the granules, which are, as we know, converted during secretion into some one or more of the organic bodies of the fluid secreted. It is somewhat in favour of this hypothesis, that in peptic glands there are apparently certain intermediate stages in the formation of pepsinogen; it may further be noted that in the liver-cells, the hyaline substance is often indistinguishably mixed with a substance allied to glycogen.

On the above hypothesis it would I think be most natural to

[1] *Arch. f. d. ges. Physiol.* xx. § 399, 1879; *Proc. Roy. Soc.* xxix. p. 377, 1879; *Journ. of Physiol.* ii. p. 261, 1879; *Trans. Roy. Soc.* Pt. iii. 1881, p. 663.

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regard the network, and peripheral layer of the cell, as the only living portions, but we have not as yet sufficient facts to allow us to come to any definite conclusion; it may be that the hyaline interfibrillar substance is protoplasmic (living) like the network, but is less differentiated. The network appears to be the result of the two-fold tendency of the protoplasm to form fibrillæ and to store up substances within its grasp; in most cases it is obviously not constant in form, but is continuously altering the arrangement of its bars and the size of its meshes. This is especially distinct in mucous cells in which during secretion numerous fresh connecting fibrillæ are formed.

(2) *Note on the Fibrin Ferment.* By A. S. Lea, M.A., and J. R. Green.

The object of this communication was to endeavour to reconcile certain statements made by Gamgee[1] as to the apparent nature and properties of the ferment body obtained from Buchanan's 'washed blood-clot' with those which are usually made respecting the properties of the ferment obtained by Schmidt's methods. From his experiments on 'washed blood-clot' Gamgee came to the conclusions that an active fibrin-ferment could be extracted from the clot, and that from the reactions of the salt solution used for its extraction the ferment "is in reality a proteid body belonging to the group of globulins." If this is so then the ferment body prepared by Gamgee must be a very different body to that obtained by Schmidt. The authors then gave an account of their own experiments which will shortly be published in detail. It will suffice here to say that by appropiate methods they were able to isolate from the 'washed blood-clot' a substance which is readily precipitable by excess of alcohol, is soluble in distilled water and does not give any reactions (Xanthoproteic) characteristic of a proteid. The aqueous solution of this substance added to a diluted salt-plasma, leads to the rapid formation of fibrin in this plasma. This power is lost if the solution of the ferment is heated for a short time to 70° C. The active properties of the solution are not very largely diminished by dialysis. From their results the authors came to the conclusion that the ferment contained in the washed clot is essentially identical with that prepared by Schmidt. They observe however that there are still some points in which it does not entirely resemble Schmidt's ferment; thus the solution of the ferment as prepared by them does not lose its activity by being dialysed (agreeing in this with Hammarsten's statements), whereas Schmidt states that his ferment solutions do largely lose

[1] *Jl. Physiol.* Vol. II. 1879, p. 145.

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their activity by dialysis. Again Schmidt says[1] that his ferment solutions, although they begin to be less active if heated to 55° C., and are materially deteriorated if heated to 70° C., require to be boiled for 5—10 minutes in order that their active properties may be entirely destroyed; the ferment body as prepared by the authors loses its active properties if heated to 70° C.

To account for the results obtained by Gamgee the authors pointed out that when 'washed blod-clot' is treated with 8 per cent. solution of sodic chloride, not only does the ferment body go into solution, but *also* a considerable proportion of ordinary globulin. This extract was apparently assumed by Gamgee to contain the ferment only and hence his results. Some other facts were also discussed in the communication, such as the apparent greater solubility of the ferment body in solutions of common salt, or as the authors prefer to consider it, the greater readiness of its extraction by such saline solutions from precipitates with which it has been carried down. Analogous cases were pointed out, such as the extraction of an amylolytic ferment from the alcoholic precipitate of liver extracts, and of 'rennet' from the mucous membrane of the calf's stomach or from the seeds of a certain plant.

The authors finally showed that ferment solutions of great activity can be prepared from ordinary fibrin, not differing so much in this respect from the similar extracts of 'washed blood-clots' as might have been expected from the statement made by Gamgee that they are 'much weaker.'

(3) *On the Structure of the Ice Plant* (Mesembryanthemum Crystallinum L.). By M. C. Potter, B.A., St Peter's College.

Mesembryanthemum Crystallinum, known as the Ice Plant, has obtained its name from the circumstance that its stem and fleshy leaves appear as if covered with minute drops of frozen water. This appearance is caused by numerous vesicles on the stem and leaves which are tensely filled with a clear colourless cell sap.

These vesicles, as described by M. Martinet[2], are of various forms, generally spherical on the upper surface of the leaf, ovoid on the lower, and elongated at the apex. Each vesicle arises as the outgrowth of a single epidermal cell, and has its apex pointed like the neck of a bottle, caused by the epidermal cell growing outwards when the vesicle is first formed, and by this part not becoming rounded off, when the vesicle afterwards increases in size.

[1] Pflüger's *Arch.* Bd. VI. (1872) S. 463.

[2] J. B. Martinet, "Organes de sécrétion des végétaux." *Ann. des Sci. Nat.*, 5th Series, Vol. XIV. 1871.

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M. Martinet has not however more fully investigated the nature and development of these vesicles.

The vesicles had been previously described by Mirbel[1] and Guettard as *Glandes utriculaires ou ampulaires*, and by Schrank as *Schlauchdrüsen*. De Candolle[2] had described them as *Glandes utriculaires* formed by a swelling of the epidermal cells and containing a colourless alkaline fluid. They had also been roughly figured by Meyen[3] and Lindley[4].

The vesicles are formed on the surface of the stem and leaves, when very young. Each vesicle arises from a single epidermal cell, and always remains unicellular. The epidermal cell about to form a vesicle grows outwards, as seen in Figures I. and IV. *a*. This part of the vesicle, which is first formed does not always alter its form, but sometimes remains as the drawn out apex of the vesicle (Fig. V.). This occurs especially on the leaves. The vesicle, when young, has its base about as wide as the cells of the ground tissue lying next to the epidermis, and which support it. As these cells increase and divide, the vesicle grows uniformly with them, and at the same time grows outwardly, forming a bladder-like excrescence on the epidermis (Figs. V. and VI.) which is supported by numerous cells belonging to the ground tissue and not to the epidermis. They are generally elongated in the direction of the growth of the part of the plant on which they are borne, and thus have their bases much longer than broad.

The vesicles formed in the manner above described are situated close to each other, are filled with a clear colourless cell sap, and so cause the plant to appear covered with ice. They are separated from each other by a small but varying number of epidermal cells, among which lie scattered numerous stomata. The vesicles are formed very early close to the growing point of the stem and branches and the leaves, whilst still very young: as these parts grow and increase in size new vesicles are formed from the young cells of the epidermis.

Thin sections of the stem or leaf-bearing vesicles stained with Chlor. Zin. Iod, (Schultz solution) shew that the wall of the vesicle is much thickened and composed of cellose, since it is stained blue; on the exterior of the vesicle, however, is seen a thin line of cuticle stained yellow, and continuous with that covering the adjacent epidermal cell. The wall of the vesicle is uniformly thickened, except that part which separates the cavity of the vesicle from the cavity of the cells supporting it. Here lenticular spaces are left unthickened, as seen in Fig. VII., a longitudinal

[1] *Elém. de Physiol. Végét. et de Bot.* Paris, 1815.
[2] *Organogr. Végét.* I. p. 78.
[3] *Ueber die Secretions-organe der Pflanzen.* Berlin, 1837.
[4] *Introduction to Botany*, Vol. I. fig. 16 *e*, page 159.

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section through the base of a vesicle situated on the stem, and in Fig. VIII. a surface view of cells supporting a vesicle[1].

Throughout their life the vesicles have a well-defined nucleus, and contain a large quantity of protoplasm. When young the protoplasm entirely fills the vesicle (Fig. IV.), but as it increases vacuoles are formed in the protoplasm, so that when the vesicle has obtained its full size a layer of protoplasm lines the inside of the vesicle in close contact with the cell wall, and stands of protoplasm proceed from this parietal layer to the nucleus (Fig. VI.). The nucleus is generally found in the protoplasm at the base of the vesicle. This protoplasm adheres more firmly to the pitted base of the vesicle, as shewn by the fact that, when the water is absorbed from the vesicle through the action of alcohol the protoplasm is found generally adhering to the basal wall and in close contact with it (Fig. V.). Where, however, the protoplasm has broken away from this wall it is found elevated and depressed so as accurately to fill the pits.

As regards the use and function of these vesicles, it would appear probable, from the fact that since the Ice Plant lives in dry sandy places, and that the exterior of the vesicles is protected by being cuticularised, that the vesicles are useful in storing up moisture during times of drought. This proves to be the case, for on an Ice Plant which had been grown in a flower pot with no water given to it, the vesicles became less and less turgid, and finally shrunk up as the soil in the pot became dry; but when the pot had been copiously watered the vesicles, after a few hours, resumed their former turgidity, and were again subjected to internal pressure. Again the vesicles are always found to be the largest when the plant is well supplied with moisture. If portions of the plant are cut off they remain green, and live for a much longer period than parts of plants which are not provided with reservoirs of moisture; the vesicles in the meantime become less and less turgid, and shrink. If, however, water is forced into the part at its cut surface the vesicles soon resume their former turgidity and are distended by internal pressure.

[1] Pits at the base of a hair are common, see Sach's *Text book of Botany*, 2nd English edition, page 43.

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(4) *On the Physiological Significance of Water Glands and Nectaries.* By WALTER GARDINER, B.A., Clare College.

Although the fact of the exudation of water and of nectar by various specialised parts of plants has been long known, the numerous observations which extend over a great number of years, deal more especially with the structural details of such organs, and, speaking generally, have not for their object the consideration of the physiological significance of the organs in question.

In the present paper I propose to briefly consider what, from the already existing literature and my own observations, would appear to be the physiological bearing of water-glands and nectaries in the general economy of the plant.

Water-glands.

It is to Unger[1], de Bary[2], and more recently to Volkens[3] that we are especially indebted, not only for their own valuable observations but also for their clear summaries of what has been made out by foregoing investigators as to the structure of water-glands, and a survey of the literature connected with the subject will be sufficient to show that our knowledge of the structural details of such organs is at the present time fairly complete.

Water-glands usually occur on the margins or at the apices of leaves though in some cases, *e.g.* species of *Crassula*, *Ficus* and *Urtica*, they may be distributed over the upper leaf surface. In structure, what is usually known as a typical gland consists of a mass of modified parenchyma cells (to which de Bary[4] gives the name of epithem cells), which abut immediately on the end of a vascular bundle; are covered externally by an epidermis; and are placed in communication with the exterior by means of one or more water-pores or water-stoma. The epithem tissue is usually distinguished by the clear hyaline character of its cells; by the abundance of protoplasm in them; and by the absence from their structure of chlorophyll grains. The vessels of the vascular bundles do not, as Volkens[5] has represented, end blindly at the epithem tissue, but between the two forms of tissue are interposed numerous tracheides, so that there is always a gradual transition from the one form to the other, and when studied from a developmental point of view[6] we see quite clearly how this transition occurs.

[1] Unger, *Beiträge zur Physiologie der Pflanzen.*
[2] De Bary, *Vergl. Anatomie*, p. 55 et seq. and 390 et seq.
[3] Volkens, *Jahrb. K. Bot. Gartens.* Berlin, 1883.
[4] De Bary, *loc. cit.* p. 391.
[5] Volkens, *loc. cit.*, see Figs. of Plates IV. V. and VI.
[6] Gardiner, *Quart. Journ. Mic. Sci.* Vol. XXI. N.S.

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Other water-glands vary from this so-called typical structure in many ways, but chiefly in the amount of epithem tissue composing the gland, and in the relative proportion that the cells of this tissue bear to those of the tracheides. Thus, as Volkens rightly observes, whereas in Dicotyledons there is usually present a well-developed gland, possessing several layers of epithem tissue interposed between the tracheides and the epidermis, in Monocotyledons the tracheides frequently end directly beneath the epidermis, or are at most, separated from it by one or two layers of cells, which never so far as could be observed, show that marked distinction from the surrounding tissue, as do the cells of the epithem tissue proper.

In Dicotyledons also great differences prevail, as for instance between the well-developed gland, such as that of *Saxifraga crustata*, with its very numerous layers of epithem, and its distinct and well-defined external sheath (see Fig. I.), and glands of a simpler structure, *e.g.* that of *Soldanella*, where the layers are few and where they are not sharply defined from the surrounding tissue.

The number of water-stoma with which such glands are provided varies within somewhat wide limits. Thus, to take one genus: *Crassula coccinea* has from 1—3, while *Crassula spathulata* has from 15—20, but they may be absent altogether, as in the glands of *Alisma* and *Sagittaria*. As noticed, especially by de Bary[1], the water-stoma differ markedly in size and appearance from the ordinary air-stoma, and as I showed in my paper "On the development of the water-gland of *Saxifraga crustata*[2]", they make their appearance long before the ordinary stomata, and their mode of formation, by simple division from a mother-cell is quite different.

Under appropriate conditions there exudes from the water-stoma, or from the free surface, of such glands as these, drops, which apparently consist practically of pure water, *e.g. Fuchsia*, &c. or water containing salts in solution—usually carbonates of Calcium and Magnesium,—as in many of the *Saxifragaceae* and *Crassulaceae*.

In the case of water-glands, as indeed in so many of the questions which concern vegetable physiology, it is to Sachs[3] that we owe by far the most important part of our knowledge as to the physiology of these organs, for he it was who clearly pointed out that the exudation of water in water-glands was dependent upon root-pressure, and was most strikingly exhibited at the period of minimum transpiration. He also found that when the activity of the roots was increased by a moderate rise of temperature, there was also an increase in the amount of the exudation. In other

[1] De Bary, *loc. cit.* p. 54 et seq.
[2] Gardiner, *loc. cit.*
[3] Sachs, *Text Book of Botany*, 1883, pp. 676 and 688.

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words, whatever increased the root-pressure increased the exudation of drops of water.

More recently there has appeared a paper by Moll[1], which also deals especially with the physiology of water-exudation by leaves. The experiments of this observer principally consisted in substituting for the normal root-pressure, that of a column of mercury, so that the amount of pressure required to bring about the exudation might both be registered and varied at will. As a whole the results obtained in this way are satisfactory, although it would appear, that this cannot be said of certain of those results, which were the outcome of very great and certainly very abnormal pressure, not only causing excessive injection of the tissue, but even in some cases a mechanical exudation of water over the whole surface of the leaf. Great interest is however attached to his observation that it is especially in young leaves that exudation most readily occurs, and that as they increase in age, exudation becomes more difficult or even impossible. Thus, with one or two exceptions, which I shall deal with later on, Moll's results confirm those of Sachs, that the exudation of water by water-glands is dependent upon root-pressure.

As regard Volkens'[2] conclusions, he finds that although in the greater number of instances, well-developed water-glands and stoma are present in the leaves of all the higher plants, yet that in certain orders *e.g. Resedaceae, Linaceae* and *Malvaceae* no such structures can be detected. In the *Papilionaceae* he makes the very interesting observation, that the stomata of the upper side of the leaf are larger in size than those of the under surface, and suggests that in this instance the exudation of water takes place by means of these slightly modified stomata. Like myself he believes that a mechanism for permitting of the escape of water is present in some form or other in all plants whatsoever, and in the light of his own, and of Moll's researches, there seems some probability, that in certain cases this function is even performed by ordinary undifferentiated stomata.

Volkens also draws attention to the difference between the glands of typical Monocotyledons and Dicotyledons. He suggests that water-glands may be regarded as the safety-valves of the plant, and that the epithem tissue acts as a resistance, put in, between the end of the vascular bundle and the free surface, so that for the expulsion of the water, through the water-stoma, a definite pressure should be required, and that the water should not merely run out, as out of a pipe.

Having thus stated, in a very brief manner, the principal conclusions with regard to the function of water-glands, I may now

[1] Moll, *Bot. Zeit.* XXXVIII, 1880, p. 25. [2] Volkens, *loc: cit.*

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deal with certain of my own results, which have some bearing on the question.

With regard to the glands of Monocotyledons and Dicotyledons, Volkens seeks in certain instances to trace a definite epithem tissue, *e.g.* in *Calla* and *Alisma*. As he rightly states, there are certain layers of cells which are marked off from the rest, both on account of their cell-contents and on account of their small and old-looking nuclei, as opposed to the large and well-developed nuclei of the neighbouring parenchyma cells, but it would seem that too much stress should not be laid upon this distinction, in consequence of the greater and more striking fact, that, stated generally, in the typical Dicotyledonous gland there is always a well-developed and very conspicuous epithem, whereas in Monocotyledons, this epithem is inconspicuous and is only recognised with difficulty. Thus in the case of *Richardia Africana* one is led to believe that the whole of the narrow projecting process of the leaf apex should be regarded as an organ for the exudation of water, and that in this organ, there is not that differentiation as in Dicotyledons, but only a number of well-developed water-stomata, and a vascular bundle which is surrounded by a few layers of practically undifferentiated parenchyma cells.

It seems an important omission on the part of Volkens[1] to leave out of his figures the tracheides, in which in all cases the vascular bundles end, for by so doing, his figure of *Alisma*, for instance, gives one the idea that the vessels of the vascular bundle end blindly, immediately beneath the epidermis, and that in older leaves, owing to the rupture of the latter tissue, the vessels open freely towards the free surface. In reality however the arrangement is like that of *Sagittaria*, which I have attempted to represent in Fig. II., and on rupture it is the tracheides which are exposed.

The water-glands and their stomata appear at a very early age in the development of the leaf. As I showed in *Saxifraga crustata*[2], the development of the leaf as a whole and of each leaf segment, is first apical and then basal. The differentiation of the young tissue into a vascular bundle and an epithem tissue rapidly proceeds, and unites the young water-gland to the general vascular-bundle system of the stem, and the whole gland is developed and in action before the appearance of a single ordinary stoma of the leaf.

As to the physiology I worked with care over Sach's results, and over certain of those of Moll. I say certain, since I preferred not to use artificial pressure, but to study the phenomena as induced by the normal pressure of the root, under varying external conditions.

[1] Volkens, *loc. cit.* [2] Gardiner, *loc. cit.*

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My results in every way confirmed Sach's statement, that the exudation in typical water-glands is dependent upon root-pressure, and does not take place from the leaves of parts of plants which have been cut off, under water, and placed under a bell-jar standing over water. Although most careful and most numerous experiments were made, I was unable to observe any exudation of water from such cut parts of plants. When however, as in many of the *Crassulaceae*, adventitious roots into the water were subsequently formed, there was at once a normal exudation.

Moll, on the other hand, maintains that in certain instances, *e.g.* species of *Impatiens* and *Fuchsia*, the exudation of water from the teeth does take place independently of root-pressure. In experimenting upon this question my results appear to show that this is not the case. With regard to *Impatiens* I find that the leaf-tooth is, as in many other instances, *e.g. Callicoma sp.* and *Fragraria vesca*, a composite structure; for the cells of the summit of the tooth are modified so as to form a nectary, while the true water-gland and water-pores are situated at its base[1]. In young leaves there does occur a secretion of drops of nectar, but the water-gland, as such, is inactive in the absence of root-pressure.

In the case of *Fuchsia globosa* some very interesting results have been obtained, for I find that if a shoot of this plant be cut off under water and placed with the usual precautions under a bell-jar, the younger leaves become bedewed with moisture, and that in some cases drops are situated at the apices of many of the leaf-teeth. The water, however, does not come from the glands but from the numerous hairs with which the surface of the leaf is freely clothed. That this is actually the case I have proved from repeated experiments which were again and again varied, the temperature being quite constant and every possible precaution being taken. The water so exuded frequently collects on the teeth, being derived from the hairs in their immediate vicinity, and gives the appearance of an exudation from the water-gland. I found in addition that the hairs of other young leaves will also exude water in the same way: the phenomenon being of course simply the expression of that very great activity, which is especially the attribute of epidermal tissue; in virtue of which the water is sucked up on one side and exuded on the other, and I have but little doubt that in certain instances this water may contain salts in solution. This will at once account for the presence of incrusta-

[1] It seems probable that the cellular bodies described by Francis Darwin on the apices of the leaflets of *Acacia sphærocephala* are of the same morphological value as these structures. See Linnæan Society Journal 1877. See also Reinke (*l. c.*).

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tions of calcic carbonate on such hairs as those of *Cheiranthus cheiri* as observed by de Bary[1].

Finally, until some further and convincing proof is brought forward, it would appear that Sach's statement still prevails, viz. that the exudation of water from the surface of water-glands is dependent upon root-pressure.

While investigating the physiology of water-glands I was struck with the fact that the exudation of water appeared to take place much more freely in darkness than when exposed to light, and in consequence I instituted a number of experiments to ascertain in how far this was actually the case. These experiments conclusively point to the fact that light retards very considerably the exudation of water, both from water-glands and from those secreting epidermal structures which are not dependent upon root-pressure, but will, like the hairs of *Fuchsia*, exude water when cut off from the parent plant. The experiments in question were made upon the following plants.

1. Water-glands:

Plants of *Saxifraga crustata*, *Saxifraga spathulata*, *Vitis antarctica*, and *Hordeum vulgare*.

2. Secreting epidermal structures:

Cut-off pieces of *Limoniastrum* (*Statice*) *monopetalum*, *Polypodium aureum*, and *Fuchsia globosa*.

1. Young plants of *Hordeum vulgare* were grown in a house, the temperature of which was about 73° F. Over some of these plants pieces of newspaper were spread, while certain of them were left uncovered. The former repeatedly exuded large drops of water; while, as regards the latter, the amount was much smaller, and in some instances there appeared to be no exudation at all.

Potted plants of *Saxifraga* and *Crassula* were covered with a bell-jar which fitted to a basin holding water. As long as they were exposed to light, the exudation was scanty, and indeed in some instances not discernible, but after having been put into a dark cupboard—the temperature remaining constant—they exuded large quantities of water, so that the leaves were bathed with the copious exudation. On coming again into light the amount exuded increased but little, and was soon evidently arrested.

A shoot of *Vitis* was placed in a manometer and exposed to a moderate mercury pressure, the shoot being under a bell-jar standing on a plate with a perforated bottom. While in the light there was no perceptible exudation, but after having been exposed for some time to darkness, large drops soon appeared on the leaf-teeth.

[1] De Bary, *loc. cit.* p. 63.

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I have only dwelt very briefly on these observations, but I would remark once for all that they were conducted with great care. The temperature was as near as possible constant, and every precaution was taken to guard against transpiration.

2. Branches of *Limoniastrum*, *Polypodium*, and *Fuchsia* were cut off under water and placed in a glass of water in a pan containing the same liquid—the whole being covered with a bell-jar. They were examined during the daytime and at night, and careful experiments were also made in the daytime, when the branches were exposed to the action of light, and when placed in artificial darkness: the temperature remaining constant. The difference in the amount exuded in light and in darkness was indeed most striking, and especially so with regard to *Limoniastrum* and *Polypodium*.

As is well known from the observations of de Bary[1] the leaves of *Limoniastrum* in common with so many of the *Plumbaginaceae* are covered with small masses of chalk. These masses occupy small depressions on the leaf-surface, and at the base of each depression is situated a group of some eight epidermal cells, which are quite unconnected with the vascular-bundle system, and, as my results show, are capable of excreting by their own activity, water containing salts in solution. The latter fact seems to me to be of some interest, and especially so, since the activity of the glands is very pronounced. For example, a piece of the stem of *Limoniastrum* was cut off and experimented upon. It was then placed in a glass of water, exposed to the air. Fourteen days after, it was placed a second time under the bell-jar and again exuded water from its glands with great vigour.

In *Polypodium* and in many of the *Filicineae* we know chiefly from the researches of Mettenius[1] that the leaves are also covered at certain areas of their surface, with little masses of chalk. Mettenius observed that these masses were placed directly over the endings of certain of the vascular bundles. The bundle apparently came to the surface, and was covered by an epidermis from which stomata were absent. Since I am not aware that there is a more detailed description than the above, I have thought it well to give a figure of the structure in question. See Fig. III. It will be seen that the bundle ends in a mass of tracheides, and is covered by some two layers of very small cells, and finally by an epidermis, with thin walls, large nuclei, and granular protoplasm. Here, as in *Limoniastrum*, there is but little doubt that it is the epidermal cells, which actively excrete the water containing chalk in solution, which, as the water evaporates, is precipitated as a thin,

[1] De Bary, *loc. cit.* p. 113.
[2] Mettenius, *Filices horti Lipsiensis.*

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but very conspicuous, incrustation over the surface of the epidermal cells.

With *Fuchsia* the same results were obtained, but in this case the activity of the cells appears to be unusually great, since they exude some water in the daytime, and when not covered with a bell-jar. This may be seen in normal, healthy, potted plants.

From these experiments there appears to be little doubt that the exudation of water, whether dependent on root-pressure or the pressure of certain localized cells, is materially influenced by light, as such, and that the action of light is unfavourable to the exudation and also retards the assumption of turgidity of cells in general. But during darkness a very definite pressure is established, which must, I think, be regarded as an important factor in determining the rapid growth which occurs at that period; at which time moreover such pressure reaches its maximum.

To return to the consideration of water-glands, it must be stated that the exudation of water from them is dependent not only upon root-pressure, but is also influenced by light.

I have now to deal in greater detail with certain other points in the structure of water-glands.

There seems but little doubt, that in all the higher plants which are provided with an apparatus for permitting the exudation of water; whether this apparatus consists of definite water-stomata or of thin walled epidermal cells, placed in connection with a vascular bundle, and whether epithem cells are present or absent; the chief function of such an apparatus is to allow of the escape of superfluous water which would otherwise cause injection of the intercellular spaces, and even rupture of the plant tissue. The presence of this excess of water in the tissue, is in the main dependent upon suspended transpiration, in consequence of which a definite and increasing root-pressure is soon apparent. The assumption of the turgid condition of the cells is also materially aided by darkness. Thus such structures are rightly to be regarded as the safety-valves of the plant. With this Moll agrees. Further, as Volkens suggests, it is extremely probable that in all plants whatsoever there exists some means of relieving excessive hydrostatic pressure. It also seems that there is little doubt that such structures are special provisions for the relieving of pressure in young organs, both from Moll's experiments as to young and old leaves, and from my own in the same direction, which certainly point to such a conclusion. This view is most remarkably confirmed by a study of the development of water-glands, since they are shown to be active at an extremely early stage in the history of the leaf.

With regard to the epithem tissue there is some difficulty. It would appear probable, as Volkens has pointed out, that in glands

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where a well-defined epithem tissue is present, such tissue may serve as a resistance, put in, between the vascular bundle and the free surface, so that the water should not mechanically run out, but that a definite pressure should be required. In connection with this subject one can but draw attention to the fact that the difference between the glands of the Dicotyledons and Monocotyledons is very remarkable, for there are in the comparative study of the glands, no gradual stages from the one group to the other, and in fact such a study seems to give some support to the view which has sometimes been put forward that the Monocotyledons and Dicotyledons are branches from a common ancestor, and that there is not a gradual ascent from the one family to the other.

I should also like to put forward here a view which has struck me, but upon which I would prefer to lay no stress. Comparing generally, the Dicotyledons with the Monocotyledons, it seems to me that whereas the former are typically land-plants in their habit, the latter on the other hand are of essentially an aquatic nature. There are, I know, many exceptions to this statement, but considering from a broad standpoint the *Gramineae*, the *Juncaceae*, *Butomaceae*, *Marantaceae*, *Zingibereaceae*, *Aroideae*, *Palmae*, and *Orchidaceae*, one finds that they are either typically water-plants or need for their growth a plentiful supply of moisture. It is of interest also to note that in most of the *Liliaceae* we have bulbs which are capable of storing up large quantities of water. Co-related to this fact we find, that as far as we know at present, we have in the Monocotyledons, glands of simple structure, and that there is little or no resistance put in, between the vascular bundle and the free surface. In Dicotyledons, on the other hand, we find, speaking generally, well-developed glands, which are present too, in plants where we should suppose that they were not needed, as in *Callitriche* as described by Borodin[1]. In the glands of Dicotyledons there is a very pronounced resistance to the escape of water, and a greater root-pressure is required to bring about exudation. The constituent cells of the plant tissue, are also rendered more turgid in consequence of the increased pressure, and the water is, so to speak, parted with much less freely. In Monocotyledons, where the supply of water is fairly constant, these special precautions for economizing water-supply are apparently not taken. With regard to the liquid exuded, we know that while the majority of these glands exude what is apparently only pure water, others exude water containing salts in solution, the salts being for the most part, carbonates of calcium and magnesium. Such salts must ob-

[1] Borodin, *Bot. Zeit.* 1869, p. 883, and 1870, p. 841. The organs of which Trecul (*Ann. de Sc. Nat.* 1854) speaks, in *Victoria regia*, I was unable to observe.

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viously be described as excreta, for so far as we know these elements are to be regarded as waste products, and are in all cases got rid of by the plant, and excluded from the general cell-metabolism, either by being excreted outside, by means of certain epidermal cells, such as those of *Cheiranthus*, *Limoniastrum* or *Polypodium*, or finally are thrown down in certain cells set apart for the purpose, as in the cells of cystoliths[1], or the crystal containing cells of so many vegetable tissues.

The glands of such plants as *Saxifraga crustata*, in their function of excretion of salts of calcium and magnesium, may well be compared to the animal kidney, and it is a matter of interest that we have in the plant as in the animal, distinct excretory glands. With regard to other epithem tissues which appear not to possess such excretory function, it is a question whether the full dignity of glands should be allowed them. For a structure to be truly glandular there must be a distinct secretion or excretion of some kind. In those cases where the epithem tissue merely plays the part of a resistance, or where, as in Monocotyledons it is scarcely developed at all, the water mechanically escapes, merely as a result of root-pressure. At present, however, it is best not to urge this point, because we have but few analyses of such exuded water, and we do not know whether pure water is exuded in every instance. On the whole, when we compare the glands of Dicotyledons with those of their Monocotyledonous neighbours, where well-developed gland cells are wanting, it would appear that the chief function of the epithem tissue is to serve as a resistance, and that in certain instances it may act as an excreting tissue[2]. We have no grounds for supposing that the general cell metabolism of Monocotyledons materially differs from that of a Dicotyledon, in virtue of which the latter should require excretory structures which the other did not possess. As regards the activity of the cells of epithems which do excrete salts, it may be remarked that although their tissues are rich in those salts in consequence of a definite excretory capacity, yet they cannot of their own activity get rid of such salts in solution, but depend upon root-pressure and other favourable conditions for the supply of a force which they cannot of themselves exert.

[1] It would be of interest to observe, whether in the case of cystolith cells there is any external exudation of water, as in *Limoniastrum*, only that in this case the salts are thrown down in the cell and do not pass through the cell walls.

[2] In those cases where an excretion of chalk takes place, crystals are not present in the tissue of the plant; where no excretion takes place, *e.g. Fuchsia* and *Urtica*, raphides or cystoliths are numerous. In *Ficus diversifolia* I did not observe cystoliths. The water-glands do not excrete salts, but the small gland on the under side of the leaf is covered by scales of what appear to be calcic carbonate. It is small in amount, however,

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As to the glands of *Limoniastrum* and *Polypodium*, they should be clearly separated from water-glands. The gland of *Polypodium* certainly resembles to some extent that of a Dicotyledonous gland, but the important distinction is, that the gland acts independently of root-pressure, and as in *Limoniastrum*, by virtue of the activity of the epidermal cells. It resembles, rather, a gland of *Limoniastrum*, which is situated immediately over the termination of a vascular bundle, and differs materially from a water-gland in the tremendous development of the tracheide tissue, as well as in its physiological properties.

One cannot but be struck by the enormous activity of the cells of the epidermal tissue, both in plants and animals. In both cases it is from the epidermal, or epithelial tissue, that well-nigh the entire secretory organs of the individual are derived. The tissue of the epithem, on the other hand, is developed not from the epidermal but from the fundamental tissue, and as far as its excretory powers go, in such glands as those of *Saxifraga*, it contrasts wonderfully with a similar structure in the animal, viz. the kidney, which is developed not from the hypoblast, but from the mesoblast, and the cells of whose urinary ducts have, apart from blood-pressure, but little excretory powers. In the plant epithem the cells go so far as to store up in their interior, large quantities of salts of Calcium and Magnesium, derived from the surrounding tissue, but they cannot of their own activity excrete and get rid of these salts. They cannot, as far as they are concerned, go so far as the excretory cells of the kidney. However, as regards the epidermal cells of the plant, attention must be drawn a second time to their wonderful activity, not only as demonstrated by root-hairs, but also, as I have shown, by the cells of the glands of *Limoniastrum* and *Polypodium*, and by the ordinary hairs on the leaves of *Fuchsia* and the like.

Nectaries[1].

Under the term nectary, which was first employed by Linnæus[2], are at the present time classed together a number of organs differing greatly from one another, both as regards their structure and their morphological value, but which are usually taken together in consequence of the fact that they secrete nectar. When we consider the tremendous differences as to morphological dignity which exist between these organs, we may well ask, as did Clos[3], whether it would not be well that the name nectary should perish

[1] The very complete papers of Martinet, *Ann. de Sc. Nat.* 5th ser. Vol. XIV. 1872, Bonnier, 6th series, Vol. VIII. 1879; and Behrens, *Flora*, 1879, give a full account of all the literature with regard to this subject, and should be consulted.

[2] Linnæus, *Syst. Naturae et Phil. Bot.* No. 86.

[3] Clos, *Ann. de Sc. Nat.* 4th ser. Vol. II. p. 23.

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from botanical terminology? It is certain that neither from their anatomical structure, nor from the appearance presented by their cell-contents, can they be distinctly recognised, and they must be regarded, as Bonnier[1] remarks, solely from their physiological point of view.

Martinet is inclined to reject a classification based upon their physiological meaning, on account of the fact that one cannot always determine by tasting, that the liquid secreted actually contains sugar; but as Bonnier points out, a delicate chemical test can always be employed, and it seems best that at present, at any rate, we should accept the suggestion laid down by Schleiden[2], to renounce all morphological distinction, and simply to regard their physiological function as Bravais[3] so long ago urged. It was a very convenient classification adopted by Caspary[4] to divide nectaries into floral and extra-floral, and in the light of this suggestion it seems better that such classification should be adopted rather than that of Mirbel[5], who separated them into vascular and non-vascular glands.

Structures capable of secreting nectar are met with on almost every part of the stem, leaf and flower. I cannot here detail the long list which would include all these various instances, but I would draw attention to the fact that they are most fully treated of in Bonnier's[6] very complete paper. To Caspary, Reinke[7], Poulsen[8], and Bonnier we owe the greater part of our knowledge with regard to extra-floral nectaries. As to floral nectaries the literature is so enormous and the investigators are so many that I cannot attempt to treat of them in the present paper, but would only draw attention to the papers of the authors I have already cited where complete and abundant references may be found[9].

The cells secreting nectar may either consist entirely of epidermal tissue, *e.g.* the nectary of the stipule of the Bean, or as in the greater number of instances epidermal and fundamental

[1] Bonnier, *loc. cit.*
[2] Schleiden, *Grundz. der wissens. Bot.* 1846, Vol. II. p. 244.
[3] Bravais, "Sur les nectaires," *Ann. de Sc. Nat.* 1842.
[4] Caspary, *De Nectariis*, Elberfeld, 1848.
[5] Mirbel, *Mém. sur l'organisat. de la fleur.* (*Mem. de l'Institut*, 1808.)
[6] Bonnier, *loc. cit.*
[7] Reinke, *Jahresbericht Bot.* 1875, p. 433. *Göttinger Nachrichten*, 1873. N. 825.
[8] Poulsen, "*Om nogle Trikomer og Nectarien.*" (*Videns. Med. K. Jöber.* 1875.)
[9] At the apex of the sepal of *Bomarea Carderi* is a structure which is apparently a nectary. I have not tested if sugar is present. The vascular bundle bends outward at this point, and lies under a mass of parenchyma, which is not well differentiated from the rest of the tissue, and is covered by an epidermis richly provided with modified stoma. This was first observed by Lynch. In the fruit of Draecena cristallina a nectary is present at the apex of each of the three ovular leaves. To Lynch is also due this observation.

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tissue. The nectary tissue may or may not be supplied with vascular bundles, and may or may not be provided with stomata.

As with water-glands, so with nectaries, the investigations extant, deal mainly with histological details; and with the exception of the observations of Sachs[1] and Wilson[2], the physiological significance of these structures has only been treated of generally, and has received but little special attention.

Sachs pointed out that the secretion of nectar was independent of root-pressure, and states that the exudation of this liquid was due to the activity of the nectary cells themselves. Wilson, on the other hand, maintains that it is due to osmosis, and not to internal pressure.

Dealing first of all with Wilson's view, it may be stated that he comes to his conclusions chiefly from the fact, that a passage of fluid from the cells of a tissue to the exterior, may be induced by placing on the surface any substance or liquid which causes osmosis, *e.g.* a drop of gum solution on a leaf of *Buxus sempervirens:* that the secretion is wholly under the control of external manipulation, for it may be stopped by repeatedly washing the nectary, and drying with blotting-paper, or induced again by the application of sugar or sugar solution: and that much collateral evidence is afforded by the consideration of what takes place in such structures as the glands of *Dionaea* which require the stimulus of some nitrogenous substance before the secretion of the digestive fluid takes place. He also draws attention to the fact that in nectaries we find that the most external walls of the nectary cells undergo a mucilaginous degeneration, and it is the mucilage thus formed, which by its osmotic properties starts the flow of nectar in the first instance. Subsequently, the flow is maintained by the osmotic activity of the nectar secreted, which is materially concentrated in consequence of gradual evaporation.

One cannot, I think, accept in their entirety Dr Wilson's views for several reasons. We have to begin with so many instances of a well-defined secretion on the part of vegetable cells, that we should expect on many grounds that the secretion of nectar was due to the activity of the constituent cells; but placing aside such considerations, one must turn especially to Dr Wilson's experiments and deductions.

It is a fact that the external walls of certain nectaries, *e.g.* those of *Nigella* and *Cestrum*, so well figured by Behrens[3], do undergo mucilaginous degeneration, but so far as I know, this cannot be taken as the universal rule, but as one is inclined to

[1] Sachs, *loc. cit.*

[2] Wilson, "The excretion of water from the surface of nectaries." *Unter. Bot. Institut*, Tübingen, 1881.

[3] Behrens, *loc. cit.* See Tab. II. Figs. 5, 6, 9 and 10.

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believe is rather the exception. Again, one cannot agree with the way in which certain of Dr Wilson's experiments were performed. Thus, it does not seem fair to make deductions as to the production of nectar flow, when the osmosis was caused by particles of sugar, which when dissolved, produced a very strong solution of great osmotic activity. We should hardly expect that this would act in the same way as a solution of a normal nectar which in some cases, *e. g. Fritillaria*, contains only 1 p. c. of sugar. As regards the constitution of nectary cells, one knows that their walls are very thin, and it seems quite possible that by the repeated washing with water a perceptible amount of sugar is removed from the cells, and that some account must be taken of the stimulus caused by the action, and the quantity of the water used in washing. We can, I think, understand that such treatment would be quite capable of seriously interfering with the normal cell equilibrium. Thus there is reason to believe that while Dr Wilson has made most valuable observations with regard to the secretion of nectar, he has not explained the cause of the secretion in the first instance. What does appear to be of special value is the great probability that the nectar at first secreted may by gradual concentration act by mere osmosis and attract more nectar from the tissue long after the cell is secreting in virtue of its own activity; and I am led to think that this goes far to explain both his own and Darwin's[1] observations, that the exudation of nectar takes place more rapidly in sunlight, for according to my own observations as regards water-glands and the like it does not seem probable that the power of secretion as such is accelerated by light.

One must conclude that the secretion of nectar in the first instance can only be explained in the light of Sach's views. We have brought before us again and again so many and so striking examples of the fact that the vegetable cell can so easily take up water on one side and exude it on the other; a fact which from some points of view might well be regarded as a property *par excellence* of living vegetable tissue; a fact moreover which in the end may be explained as the effect of a somewhat complex physical law, but of which we have at present no explanation. Most striking examples of this force are afforded by root-hairs, and by the epidermal cells of such glands as those of *Limoniastrum*. In fungi, *e. g.* the unicellular *Mucor* in *Pilobolus*, in the root-hairs of *Marchantia*, and finally, as Sach has shown, in pieces of the cut stems of young grasses when placed in damp earth the same phenomena prevail.

In fact, there can be but little doubt that in nectaries the first

[1] Darwin, *Cross and Self-fertilisation of Plants*, p. 403.

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secretion is due to the fact that the secreting cells "absorb water (containing substances in solution) with great force on one side and exude it on the other[1]", and this solely by their own living activity. The grand period of such activity may be and probably is of short duration, and the further exudation of nectar may be occasioned in the way that Wilson has pointed out, viz. by simple osmosis. In such cells as those of *Limoniastrum*, on the other hand, this grand period of activity is, as I have shown, of long duration.

It will be observed that I have spoken of the excretion of salts of calcium and the secretion of nectar. I did so on the ground that an excretion is a substance which is of no further use to the organism, whereas from the fact that the nectar contains sugar, and as first remarked by Bravais, and confirmed by Bonnier and Wilson, is frequently reabsorbed into the tissue, there seems but little doubt that the sugar of nectar can be used up in the vegetable metabolism, and must therefore be regarded as a secretion.

The view as to the use of nectar in the plant economy is, that it is for the purpose of protection in the case of certain of the extra-floral nectaries, and to insure fertilisation by attracting insects, as in the nectaries of flowers, and also in the extra-floral structures, such as those of *Marcgravia*. It must be confessed, however, that at present we cannot give an entirely satisfactory account of the part played by nectar in the plant economy, and this is especially true as regards extra-floral nectaries.

In concluding, I beg to express my thanks to Mr R. I. Lynch, the able curator of the University Botanic Gardens. He has repeatedly made for me valuable observations, and many of the experiments with regard to the effect of light upon the exudation of water were made for me by him, and always with great success. His observations with regard to the occurrence of nectaries in the sepal and in the fruit are both new, and of great interest.

Explanation of figures illustrating Mr Gardiner's paper "On the Physiological Significance of Water-glands and Nectaries."

Fig. I. Longitudinal section through a fully-developed gland of *Saxifraga crustata* (after Gardiner).

w = water stoma.
e = epithem tissue.
t = tracheide tissue.
v = vascular bundle.
h = hairs to catch the deposited calcic carbonate and prevent it from stopping up the water pore.

[1] Sachs, *Text-Book*, p. 688.

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Fig. II. Longitudinal section of the apex of the leaf of *Sagittaria Monte Vidiensis*.

v = vascular bundle.
t = tracheide tissue.
e = cells corresponding to the epithem tissue of Dicotyledons.

Fig. III. Longitudinal section of a gland of *Polypodium aureum*.

v = vascular bundle.
t = tracheide tissue.
e = excreting epidermis.

Fig. IV. Longitudinal section of the nectary of the cotyledon of *Ricinus communis* (after Bonnier).

v = vascular bundle.
s = surface of nectary.

Fig. V. Longitudinal section through the spur at the base of the petal of *Nigella arvensis* (after Behrens).

v = vascular bundle.
n = nectary surface.

November 26, 1883.

Mr Glaisher, President, in the Chair.

The following were duly balloted for and elected Fellows of the Society:—

Prof. Macalister, M.A., F.R.S., St John's College.
Mr J. C. McConnel, B.A., Clare College.
Mr E. J. C. Love, B.A., St John's College.

The following communications were made to the Society:

(1) *On the measurement of electric currents.* By Lord Rayleigh, M.A., F.R.S.

Perhaps the simplest way of measuring a current of moderate intensity when once the electro-chemical equivalent of silver is known, is to determine the quantity of metal thrown down by the current in a given time in a silver voltameter. According to Kohlrausch the electro-chemical equivalent of silver is in C. G. S. measure $1{\cdot}136 \times 10^{-2}$, and according to Mascart $1{\cdot}124 \times 10^{-2}$. Experiments conducted in the Cavendish Laboratory during the past year by a method of current weighing described in the British Association Report for 1882 have led to a lower number, viz. $1{\cdot}119 \times 10^{-2}$. At this rate the silver deposited per ampere per *hour* is 4·028 grams, and the method of measurement founded

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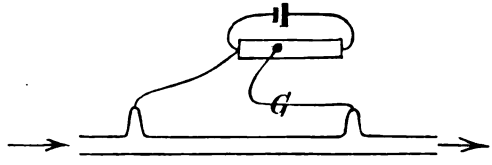
upon this number may be used with good effect when the strength of the current ranges from $\frac{1}{20}$ ampere to perhaps 4 amperes. It requires however a pretty good balance, and some experience in chemical manipulation.

Another method which gives good results and requires only apparatus familiar to the electrician, depends upon the use of a standard galvanic cell. The current from this cell is passed through a high resistance, such as 10,000 ohms, and a known fraction of the electro-motive force is taken by touching this circuit at definite points. The current to be measured is caused to flow along a strip of sheet German silver, from which two tongues project. The difference of potential at these tongues is the product of the resistance included between them and of the current to be measured, and it is balanced by a fraction of the



known electro-motive force of the standard cell (Fig. 1). With a sensitive galvanometer the balance may be adjusted to about $\frac{1}{4000}$. The German silver strip must be large enough to avoid heating. The resistance between the tongues may be $\frac{1}{200}$ ohm, and may be determined by a method similar to that of Matthiessen and Hockin (Maxwell's *Electricity*, § 352). The proportions above mentioned are suitable for the measurement of such currents as 10 amperes.

Another method, available with the strong currents which are now common, depends upon Faraday's discovery of the rotation of the plane of polarization by magnetic force. Gordon found 15°[1] as the rotation due to the reversal of a current of 4 amperes circulating about 1000 times round a column of bisulphide of carbon. With heavy glass, which is more convenient in ordinary use, the rotation is somewhat greater. With a coil of 100 windings we should obtain 15° degrees with a current of 40 amperes; and this rotation may easily be tripled by causing the light to traverse the column three times, or what is desirable with so strong a current, the thickness of the wire may be increased and the number of windings reduced. With the best optical arrangements the rotation

[1] [Jan. 1884. In a note recently communicated to the Royal Society (*Proceedings*, Nov. 15, 1883) Mr Gordon points out that owing to an error in reduction, the number given by him for the value of Verdet's constant is twice as great as it should be. The rotations above mentioned must therefore be halved, a correction which diminishes materially the prospect of constructing a useful instrument upon this principle.]

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can be determined to one or two minutes, but in an instrument intended for practical use such a degree of delicacy is not available. One difficulty arises from the depolarizing properties of most specimens of heavy glass. Arrangements are in progress for a redetermination of the rotation in bisulphide of carbon.

(2) *On the measurement of Temperature by Water-vapour pressure.* By W. N. SHAW, M.A.

[ABSTRACT.]

The method practically adopted for accurately measuring temperature is to use a mercury thermometer as a thermoscope and express its indications in terms of some accepted standard of temperature by direct or indirect comparison.

The methods of comparison usually employed are by the air-thermometer, by calibration and repeated determinations of the fixed points and the method of direct comparison adopted at the Kew Observatory. The last is the one which is most generally applicable, but it necessitates the sending of the instrument to Kew, and examples of its application shew that it cannot be relied upon to give greater accuracy than a tenth of a degree.

A water-steam thermometer, as suggested by Sir W. Thomson, (*Ency. Brit.* Edit. IX., Art. 'Heat') may be used to give a standard scale of temperature and mercury thermometers compared with it; and instead of measuring directly the saturation-pressure e of water-vapour, it can be calculated by the formula

$$\frac{e}{760} = \frac{1+\alpha t}{\Delta} \cdot \frac{f}{d},$$

from an observation of the weight of water contained in a known volume of air artificially saturated, where f is the quantity of moisture per unit volume of the air, d the specific gravity of water-vapour referred to air at the same temperature and pressure, Δ the density of air at 0°C. and 760 mm. pressure, and t the temperature of the air (which may be measured by an uncorrected thermometer) and α the coefficient of expansion. The temperature is then accurately given by Regnault's table of tensions of water-vapour at different temperatures and thus the indication of the thermometer to be tested, placed in the saturating space, can be corrected.

The quantities Δ and α of the formula are known constants, the value of d was shewn by Regnault's experiments (*Ann. d. Chim.* [3] XV.) to be very nearly constant and equal to ·622.

The experimental part of the comparison of any thermometer with a water-steam thermometer by this method consists in the determination of f. This requires only the same experimental

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arrangement as that for the determination of the tension of aqueous vapour in the air, by the chemical method of absorption of the moisture by desiccating substances, with the addition of an apparatus for previously saturating the air.

The ordinary form of the apparatus is modified by avoiding as far as possible india-rubber connections, using glass tubes and mercury joints for making communication between the different parts. The saturating apparatus is similar to that used by Regnault.

The desiccating substances must be either sulphuric acid or phosphoric anhydride; chloride of calcium does not completely dry the air passed over it.

With such an apparatus, which is easily constructed, corrections for a thermometer enclosed in the saturating vessel can be determined which, for the thermometers experimented on, agree with the Kew corrections to within a tenth of a degree.

(3) *Measurement of the Dark Rings in Quartz.* By J. C. McConnel, B.A.

MacCullagh has explained the peculiar optical behaviour of quartz, by introducing into the equations of motion of light terms involving differential coefficients of the third order. These equations contain in addition to the two principal wave velocities only one arbitrary constant. So it becomes of interest to see if all the facts are really accurately contained in this simple theory. I believe the only published observations on the subject are those of Jamin in 1850. He made three series of experiments on plates of quartz cut perpendicular to the axis, using sodium light. In each case he employed parallel rays. In the first series he used a Babinet's compensator to observe the nature of the light originally plane polarised, which had passed through a plate of quartz. By this means he obtained values of the retardation of one ray in the quartz relative to the other, and of the ratio of the axes of the ellipses of polarisation of both rays. In the second series he inclined a plate of quartz between two crossed Nicols and observed the angles when the transmitted light was completely quenched. This would happen whenever the relative retardation was an exact number of wave lengths. In the third series he examined the nature of the transmitted light at those points in the preceding experiment where it reached a maximum. He found that his observations confirmed the theory within the limits of experimental error. But these limits were not very close, as the average differences between the observed and the calculated values were about $\frac{1}{40}$th of the quantities to be determined and there were several wide discrepancies; one or two amounting to $\frac{1}{12}$th.

I thought that I would try if a more accurate determination of

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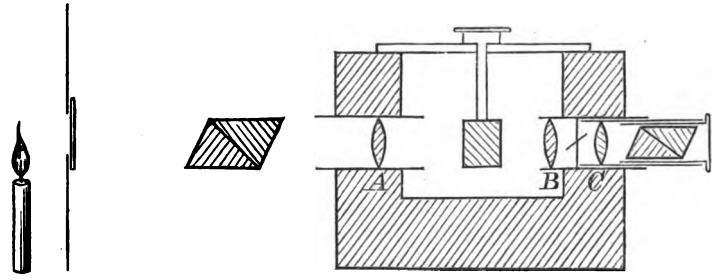
the relative retardations could not be obtained by measuring the well-known dark rings seen when a plate of quartz is inserted between two crossed Nicols; the light being made to pass through the quartz in a convergent pencil. For this purpose Professor Lewis was kind enough to lend me a polariscope and some thick crystals of quartz cut perpendicular to the axis.

It is important that the plate of quartz should be thick; for the peculiarities of quartz are only manifested when the light makes a small angle with the axis, and the thicker the plate the more rings will be formed within this small angle. The plate of quartz I used was about an inch thick.

The arrangement of apparatus was as follows. First a sodium flame placed behind a screen with a small aperture which was closed with ground glass. This was found to give the necessary steadiness to the light. At a distance of about four feet was placed a Nicol's prism. Then came the frame of the polariscope which supported the rest of the apparatus. This consisted of a lens *A* to make the light converge; the crystal of quartz; a lens *B* at whose



focus the cross wires were put; an eye lens *C* which rendered the rays parallel; and a second Nicol next the eye. The quartz was attached to a horizontal circle on the top of the frame, fitted with a vernier reading to minutes. This afforded the means of measuring the diameters of the rings. The lens *B* was limited by a diaphragm with a small aperture; so as to secure that the pencil of light should pass through the centre of the lens, and thus errors, due to spherical aberration and the cross wires not being accurately in the axis of the lens, should be avoided. When the quartz was inclined the lateral displacement it produced on the light was so great that it was necessary to also incline the frame of the polariscope to the incident light; but this had no influence on the readings. The cross wires were really thin lines ruled on glass and blackened.

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After some preliminary trials the first careful set of observations were taken by turning the quartz till the thin line of light between the ring and the cross wire almost disappeared first on one side, then on the other, and taking the mean of the two readings. I took two series of these, and the two means agreed with one another usually within one minute, though in one case they differed by about six minutes. To get the radii I found the centre of each ring, and then measured the radius of each ring to right and to left from the mean centre. The general result was that the observed radius of the second ring was 5′ greater than the calculated; of the third ring 3′ greater; while the other rings agreed within 2′.

I then took a similar double set of observations of the second and third rings, and measured the first ring by placing the cross wire over the blackest part. These observations agreed pretty well with one another and with the previous ones, the extreme differences amounting to 3′. The observed radius of the first ring was 10′ greater than the calculated.

Next I took a double set of observations of the first nine rings; this time placing the cross wire as nearly as possible in the centre of each black band. These observations were not quite so good, the extreme differences amounting to 4′, and in the case of the first ring to 7′. Still the mean results agreed with the previous ones within 3′.

The two methods of reading just described have a tendency to give rather too small values of the radii of the rings. This is due to the fact that the rings shade off more rapidly on the outside than the inside. What we actually measure is the middle of the dark band; while the darkest part of the band, which is what we want to measure, lies on the outside of the middle. This error may be estimated in the above observations to amount to about 5′ in the second ring and 2′ in the third ring. So if we take this into account we double the discrepancy between the observed and the calculated values.

I thought it possible however that there might be some error, due to the difficulty of measuring a curved band with a straight cross wire; so I took out the cross wires and substituted for them two needle points opposite one another and nearly in contact. I put the right hand needle point in contact with the right side of the band, and then the left needle point in contact with the other side, and took the mean of the two readings for the middle of the band. These observations, though not quite so accurate as the previous ones, confirm their results and shew that no serious error was due to the curvature of the bands.

Besides these I took some observations with the needle points more widely separated, so that I could place the band exactly

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between them. These however, since the points were so far apart, were still more subject to the error I have described. Indeed it would probably amount to 10′ in the second ring and 5′ in the third. Taking this into consideration, they agree fairly well with the former observations. But as it is rather a haphazard correc-

TABLE I.

No. of Obs.	Ring No.	1	2	3	4	5	6	7	8	9	10
	4	.....	7° 46½′	9° 28′	10° 38′	11° 43′	12° 38½′	13° 31′	14° 20′	15° 5′	15° 46½′
	4	4 22	7 47½	9 26	.....	.....	.....	.....	.....	.....	.....
	4	4 22½	7 46	9 25	10 38½	11 43	12 40	13 33	14 20	15 5½	15 48
	2	.....	7 47	9 21½	10 38½	11 42½	12 38	13 33	.....	.....	.....
	Mean of 14	4 22	7 47	9 24	10 38½	11 43	12 39	13 32	14 20	15 5	15 47
	Prob. + correction	?	5′	2′	1′	½′	½′	½′	½′	½′	½′
	Corrected Obs. Val.	?	7 52	9 26	10 39½	11 43½′	12 39½	13 32½	14 20½	15 5½	15 47½
	Calculated Val.	4° 12½	7 41	9 20	10 36	11 40	12 37	13 30½	14 19	15 5	15 48½
	Difference	+10′?	+11′	+6′	+3½′	+3½′	+2½′	+2′	+½′	+½′	−1′

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tion, I have not thought it advisable to apply it or to introduce the observations without it.

The results of my observations are stated in the table opposite. The first column gives the number of determinations of which the given figures are the mean. I have then taken the mean of the observed values giving equal weight to all fourteen observations. I have added the probable corrections already mentioned, and then taken the difference between these corrected observed values and the calculated values.

These probable corrections are obtained in the following manner. In the observations with the needle points the difference between the readings for the two sides of a band varies from about 27′ for the second ring to about 17′ for the seventh ring, while in the observations with the cross wire the difference falls from 36′ to 27′.

It is clear that in one case the distance between the needle points has to be added in order to obtain the apparent breadth of a band, and in the other case the thickness of the cross wire has to be subtracted. The thickness of the cross wire was very considerably greater than the distance between the needle points; so we shall not be far wrong in putting the apparent breadth of the second band at 29′. Dividing this in the ratio of the calculated distances of the two adjacent bands we get an approximate value of the position of the darkest part of the band. Of course the apparent breadth of a band varies to some extent with the brightness of the light, but by taking the mean breadth we get the proper correction to apply to the mean observed value. The apparent breadth of the third band was similarly found to be about 24′ and so on to about 20′ for the seventh band. The first band was so much fainter and more ill-defined that the above remarks do not apply and the proper correction is probably small.

With regard to the calculation. MacCullach (*Trans. Roy. Ir. Acad.* 1837, p. 463) assumes the equations

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= A\frac{d^2\xi}{dz^2} + C\frac{d^2\eta}{dz^2} \\ \frac{d^2\eta}{dt^2} &= B\frac{d^2\eta}{dz^2} - C\frac{d^2\xi}{dz^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

Thus ξ and η are the displacements parallel to x and y at any time t. The axis of z is the wave normal.

$$\left.\begin{aligned} A &= a^2 \\ B &= a^2\cos^2\phi + b^2\sin^2\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where a and b are the principal wave velocities and ϕ is the angle between the wave normal and the optic axis.

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The principal vibrations are given by

$$\left.\begin{aligned}\xi &= p\cos\frac{2\pi}{l}(st-z)\\ \eta &= q\sin\frac{2\pi}{l}(st-z)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

when p, q, s, l are constants connected by the relations

$$\left.\begin{aligned}s^2 &= A-\frac{2\pi}{lp}Cq\\ s^2 &= B-\frac{2\pi}{lq}Cp\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From these we get $(s^2-A)(s^2-B)=\dfrac{4\pi^2C^2}{l^2}$ .....(5).

From the formula (3) it is clear that s is the velocity of a wave whose wave length is l. If the wave length of the same light in air is λ, then $l=s\lambda$. But as l only appears in the second term of (5) which we shall find is very small, we may put $l=a\lambda$. Thus

$$(s^2-A)(s^2-B)=\frac{4\pi^2C^2}{a^2\lambda^2}\quad\ldots\ldots\ldots\ldots\ldots(6).$$

Solving,

$$s^2=\frac{A+B}{2}\pm\frac{\sqrt{(A-B)^2+\dfrac{16\pi^2C^2}{a^2\lambda^2}}}{2}\quad\ldots\ldots\ldots\ldots(7).$$

Let s_1, s_2 be the two values of s. Then rejecting squares of the third term in the above equation

$$\frac{1}{s_1}-\frac{1}{s_2}=\frac{\sqrt{2}\sqrt{(A-B)^2+\dfrac{16\pi^2C^2}{a^2\lambda^2}}}{(A+B)^{\frac{1}{2}}}.$$

If L be the length of the crystal traversed, measured normal to the wave front, $L\left(\dfrac{1}{s_1}-\dfrac{1}{s_2}\right)$ is the difference of times of traversing the crystal, and is therefore the relative retardation in air, the velocity in air being unity. Denoting the relative retardation by R we have approximately

$$\frac{R^2}{L^2}=\frac{(a^2-b^2)^2\sin^4\phi+\dfrac{16\pi^2C^2}{a^2\lambda^2}}{4a^6}\quad\ldots\ldots\ldots\ldots(8).$$

In the above we have made four approximations; puttting a for s in (6); rejecting squares of the third term in (7); tacitly

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assuming ϕ is the same for both waves; and putting $A+B=2a^2$ in (8). It may be shewn however that no one of these approximations can produce an error of as much as half a minute in the present case.

If we put $\phi=0$ we obtain

$$\frac{R}{L}=\frac{2\pi C}{a^4\lambda},$$

and it is not difficult to see that $\rho/180=R/\lambda$ where ρ is the rotation measured in degrees. Hence $\rho/180=2\pi CL/a^4\lambda^2$.

The above calculation is taken partly from MacCullagh and partly from Verdet. I have thought it better to give it, as MacCullagh does not give equation (8) and Verdet's result is affected by a slight numerical error. I have used MacCullagh's notation.

If in the above formula (8) we put $R=n\lambda$ where n is an integer and insert the known values of the constants we get a series of values of ϕ corresponding to the dark rings. Then the formula $\sin\phi=a\sin\phi'$ gives the value of ϕ' which is the angular radius of the ring outside the quartz. The first ring is given by $n=3$.

The thickness T of the piece of quartz I used was by a careful measure found to be 24·00 mm. and $L=T\sec\phi$. The value of C is given on MacCullagh's theory by the formula $2\pi C/a^4\lambda^2=\rho/180$ where ρ is the rotation, measured in degrees, due to a plate of quartz of unit thickness. Brock has found the rotation for a millimetre to be 21·67°. He experimented on several different pieces of quartz, and found the mean differences to be about a tenth of a degree.

I have used the values of $1/a$ and $1/b$ given by Rudberg 1·54418 and 1·55328.

In different specimens of quartz they apparently vary about ·0001. Thus $a-b$ might vary about 1/60 of itself which would make a difference of about 8′ in the tenth ring. The values I have taken are confirmed by the agreement between the observed and the calculated radii in the larger rings.

In the tenth ring the absence of C would only make a difference of 12′, while it would almost double the first ring. The observed and the calculated values of the radii would be reconciled in the case of the second ring by a change in C of 1/10th of its value and in the case of the third ring of 1/7th; while a much smaller change would suffice in the case of the first ring.

The final formula from which the values of ϕ' were calculated was

$$\sin^4\phi'=\cdot00004207\,n^2\cos^2\phi-\cdot0003487\ldots\ldots\ldots(9).$$

An approximate value of ϕ is sufficient in the second term.

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The following table gives the changes in the retardation to which the differences recorded in the last line of Table I. correspond.

TABLE II.

Ring No.	1	2	3	4	5	6	7	8	9	10
ϕ	2° 46′	5° 8′	6° 12′	7° 2′	7° 48′	8° 24′	9° 0′	9° 32′	10° 3′	10° 32′
$-\frac{\delta R}{R}$	·005 ?	·024	·015	·008	·007	·004	·003	·002	·0005	- ·0015

The second line contains approximate values of the angle between the wave normal and the optic axis; while the last line

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gives the ratio of the difference between theory and observation to the whole relative retardation. Thus the general result I have obtained is that MacCullagh's theory gives values of the relative retardation which are slightly too large. The differences in the last three or four rings may be fairly put down to errors of observation.

I have said nothing about the surface effects. If the plate had been isotropic, the plane of polarisation would have been turned a little when the light entered the crystal and again turned a little in the same direction when it left it. If θ be the angle between the plane of polarisation of the incident light and the plane of incidence, and θ' be a similar angle with regard to the emergent light, then $\tan\theta' = \tan\theta \sec^2(\phi' - \phi)$. This gives $\theta' - \theta = 12'$ at the most in the case of the 10th ring. Now it requires a rotation of either Nicol through two right angles to move a dark band from its place as one ring to its place as the next, so this small change in θ would have no appreciable influence. We may assume, I think, that the surface effects in quartz are of something the same nature and magnitude as in an isotropic substance.

The experiments were made in the Cavendish laboratory in August and October. I am much indebted to Mr Glazebrook for his kind assistance and advice throughout.

(4) *On the origin of segmented animals and the relation of the mouth and anus to the mouth of the Cœlenterata. By* ADAM SEDGWICK, M.A.

In the discussion which followed the communication of the late Professor Balfour's notes and drawings of the early embryos of *Peripatus Capensis* to the Royal Society (December, 1882) I drew attention to the great resemblance between the embryo of P. Capensis with its elongated blastopore and somites, and an adult Actinozooid Polyp. I pointed out that the comparison of these two structures suggested an explanation which, so far as I know, has not before been suggested, of a great morphological difficulty, viz. the origin of metameric segmentation (vide *Nature*, Dec. 28, 1883). At the same time I pointed out that by following up this comparison some other morphological difficulties received an explanation. The hypotheses[1] I suggested were shortly as follows.

[1] Mr E. B. Wilson who was present when this discussion took place at the Royal Society and to whom I subsequently at Cambridge showed the specimens and drawings of the Peripatus embryo, informs me that the work (referring to Polyps) which he has since done at Naples has enabled him to give some additional evidence in favour of my views. As Mr Wilson's observations are not yet published, I am

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1. The **mouth** and **anus** found in most of the higher groups (*Vermes, Mollusca, Arthropoda*, and in all probability *Vertebrata*) have been derived from the mouth of an ancestor common to them and the Cœlenterata, i.e. from an elongated opening such as is found at the present day in the Actinozoa.

2. That the **somites** of segmented animals are derived from a series of pouches of the primitive gut (archenteron) of a Cœlenterate-like ancestor, i.e. from pouches generally resembling those found at the present day in Actinozooid Polyps and Medusæ.

3. That the **excretory organs** or nephridia (segmental organs) of the higher animals are derived from specialized parts of these pouches which were in the supposed ancestor, as indeed they now are in many living *Medusæ* and *Actinozoid Polyps* connected peripherally with each other by a longitudinal canal (circular canal of *Medusæ*, perforations in mesenteries of *Actinozoa*) and with the exterior by a pore[1], one for each pouch: further that in the *Invertebrata*, e.g. *Annelida* the longitudinal canal has been lost and the external pores retained, while in the *Vertebrata* the longitudinal canal has persisted (*segmental* or *pronephric duct*) and retained its posterior opening into the alimentary canal while the external pores have been lost.

The essence of these three propositions lies in the fact that the segmented animals are traced back, not to a triploblastic unsegmented ancestor but to a two-layered Cœlenterate-like animal with a pouched gut, the pouching having arisen as a result of the necessity for an increase in the extent of the vegetative surfaces in a rapidly enlarging animal (for circulation and nutrition).

The hypotheses are based upon the embryonic development of the respective organs in the Triploblastica, and the structure of the living Cœlenterata; in other words upon facts precisely of the same nature as those which have been used in tracing the evolution of the nervous and muscular tissues.

Before proceeding to summarise the facts upon which the hypotheses rest, I may be permitted again to point out that it is no part of my view to derive segmented animals direct from the Cœlenterata, but to derive both *Cœlenterata* and *segmented animals* from a common Cœlenterate-like ancestor whose structure can only be elucidated by studying the anatomy and the development of the living Cœlenterates and of the higher segmented animals. The main facts are shortly as follows.

unable to quote them here, but he informs me that his paper is in the press and will shortly appear in the Naples "Mittheilungen."

[1] Vide Hertwig, "Organismus der Medusen," p. 39; and "Die Actinien," *Jena. Zeitschrift*, Bd. XIII.

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Facts of Cœlenterate Anatomy.	Facts from development of Triploblastica.
1.	1.
Elongated mouth of *Actinozoa*, and the differentiation of that mouth into two parts[1].	The embryonic development of *Peripatus Capensis*[2].
The special aggregation of the nervous system round that mouth (*Medusæ*, *Actinozooid Polyps*).	The slit-like form of the blastopore in several animals (first pointed out by Lankester in Mollusca).
Bilateral symmetry of Actinozoa.	The peculiar behaviour of the blastopore in several animals receives light from this hypothesis.
	The arrangement of the central nervous system in the Triploblastica.
2.	2.
The Cœlenterata present a serial repetition of certain organs similar to that found in metamerically segmented animals; and just as in the latter the repetition corresponds in each case to the mesoblastic somites, so in the former it corresponds with that of the alimentary pouches. Thus for each mesoblastic somite we find typically in segmented animals a nerve ganglion, a nephridium, a segment of the muscular system, and an external appendage; while for each pouch in a *Medusa* we find a sense organ or ganglion or both, an excretory pore (Hertwig *loc. cit.*), a segment of the muscular system (circular striated muscles of subumbrella of *Medusa* broken up into segments by each radial pouch) and a tentacular organ.	The development of the body cavity from archenteric pouches in *Brachiopoda*, *Sagitta*, *Balanoglossus*; and particularly in *Amphioxus* in which each somite (at least the 14 anterior pairs) is derived from a distinct pouch.

[1] Vide Hertwig, "Die Actinien," and Hickson, "On the Ciliated Groove (Siphonoglyphe) in the Stomodæum of Alcyonarians," *Proc. Roy. Soc.* no. 226, 1883.

[2] Balfour, "Anatomy and development of Peripatus Capensis," *Quart. J. of Mic. Sci.* April, 1883. As Dr v. Kennel has chosen to throw some doubt upon the correctness of the observations recorded in this paper, I will take this opportunity of stating that the facts enumerated on pp. 256, 257 which Professor Mosely and myself regarded as established by the investigation of the embryos then at our disposal, are perfectly correct. I may perhaps be permitted to point out to Dr v. Kennel a fact which he has apparently overlooked, viz. that he is working at *P. Edwardsii*, a species differing considerably from *P. Capensis*.

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3.	3.
Marginal pores in *Medusæ*.	Development of excretory organs in Elasmobranchs and other Vertebrates[1].
Openings in body wall of *Actinia*.	Development of excretory organs in *Polygordius* (Hatschek).

To sum up in a few words: the Cœlenterata differ from segmented animals only in the fact that the alimentary or archenteric pouches (mesoblastic somites) and the alimentary canal do not become separate, and connected with this absence of a distinct cœlom is the low state of differentiation of such cœlomic structures as the excretory organs and the absence of a separate vascular system.

In conclusion I may say that I do not put forward these hypotheses in a dogmatic spirit, and that I fully recognize that theories dealing with the complicated facts of Morphology can only have in most cases a very temporary value. The hypotheses just discussed first occurred to me some years ago when investigating the development of the vertebrate excretory organs, and they have received such striking confirmation from Hatschek's work on Amphioxus and more recently from the embryo of Peripatus Capensis that I have at length decided to publish them, hoping that they may at least excite criticism, and so lead to the increase of our knowledge and to the greater definition of our ideas.

A full discussion of the facts bearing upon the views advocated in this paper, illustrated by explanatory diagrams, together with some speculations (following the same lines) on the origin of tracheæ and gill slits, and on the evolution of the vertebrata, will I trust appear in the January number of the *Quarterly Journal of Mic. Science.*

[1] Vide Sedgwick, "Development of Kidney, etc." *Quart. J. Mic. Sci.* Vol. xx. 1880, and "Early development of Wolffian duct, etc." *ibid.* Vol. xxi. 1881.

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EXPLANATION OF FIGURES.

Plate II.

Figs. I and IV *a*. First indication of a vesicle shown by the protuberance of a single epidermal cell.

Figs. II, III, and IV, Successive stages in the development of the vesicle.

Fig. V, Mature vesicle from a leaf showing the drawn out apex of vesicle; seen in transverse section.

Fig. VI, Mature vesicle from stem.

Fig. VII, Longitudinal section through the base of a vesicle on stem, showing the pits at its base.

Fig. VIII, Pits on the base of a vesicle, seen from above.

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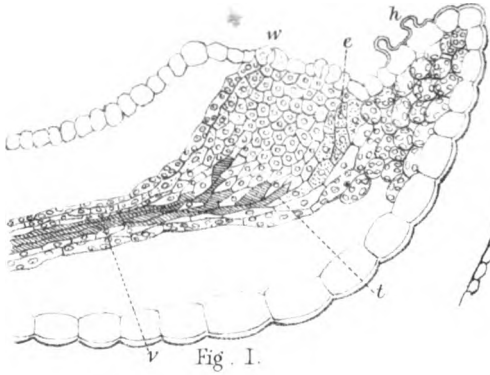

Fig. I.

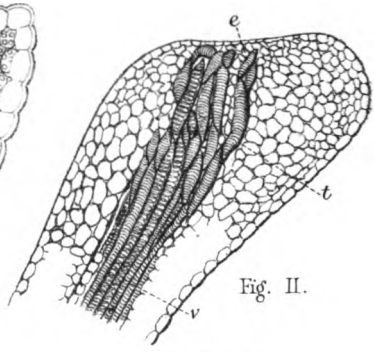

Fig. II.

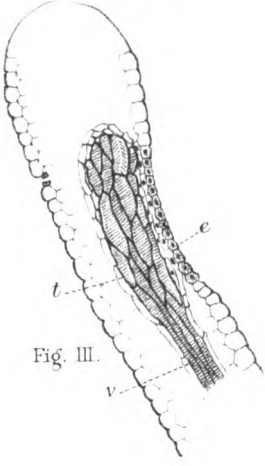

Fig. III.

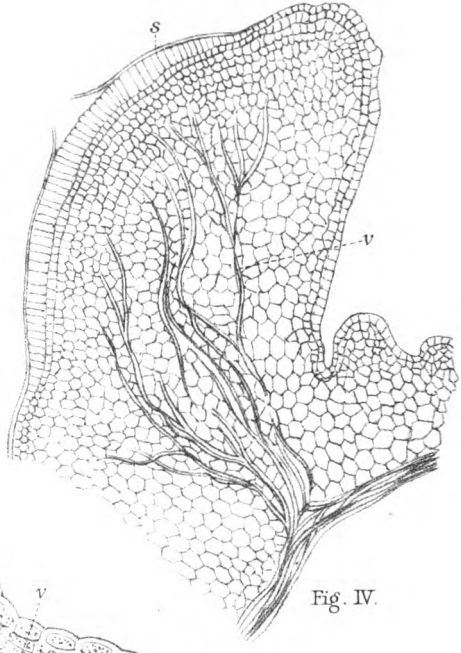

Fig. IV.

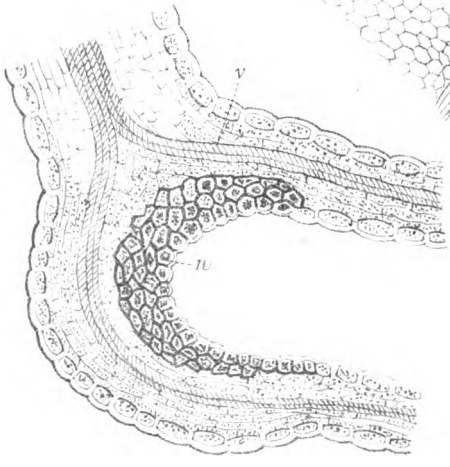

Fig. V.

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Fig. I.

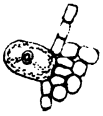

Fig. II.

Fig. III.

Fig. IV.

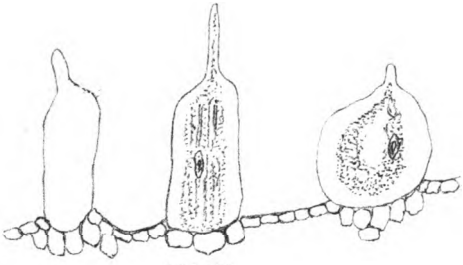

Fig. V.

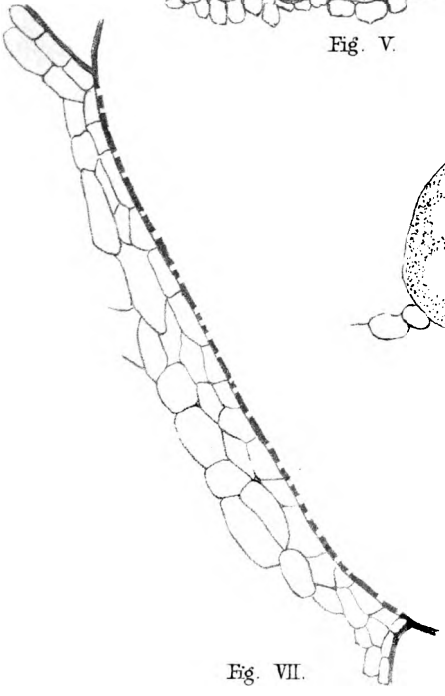

Fig. VII.

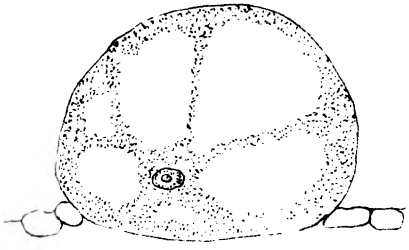

Fig. VI.

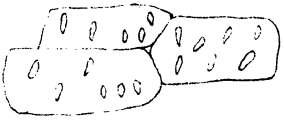

Fig. VIII.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

January 28, 1884.

Mr Glaisher, President, in the chair.

The following communications were made to the Society:

(1) *On the Microscopic structure of a Boulder from the Cambridge Greensand found at Ashwell, Herts.* By Prof. T. G. Bonney, D.Sc., F.R.S., F.G.S.

This boulder was found by H. G. Fordham, Esq., F.G.S., by whom a fragment was sent to the author for examination, with the following remarks.

"The boulder measures $12 \times 9\frac{1}{2} \times 5\frac{1}{2}$ inches, and is therefore amongst the largest at present known from this bed. It is somewhat triangular in general form, one surface being nearly flat, and it is very much rounded and worn. On the weathered surface dark purple wavy lines appear, generally of the thickness of a sheet of writing paper, but sometimes a quarter, or even half an inch thick, alternating with lighter and thicker bands. Where broken the rock is more uniform in colour, the bands varying in shades of purple. Occasionally, where much weathered, the lighter bands show a tendency to columnar structure, developed perpendicularly to the planes of banding. The material is very hard, and not easily broken. The surface of the boulder is worn and smoothed, and in some parts may almost be said to be polished. Here and there the softer material of the light-coloured bands has been worn into small cavities or depressions, and in other places the lines of banding are brought into strong relief by a more uniform wearing away of the softer bands.

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"As is usually the case with the boulders and fossil remains of the Cambridge Greensand, this specimen has upon its surface a number of attached plicatulæ and other small shells, and it bears also two patches of the phosphatic nodules characteristic of the bed from which it has been obtained, and even a fragment of the marl itself.

"While the boulder has clearly been subjected to very great wear, and has the external appearance usually attributed to the action of ice when found in similar boulders of more recent periods, there are upon it no distinct or definite scratches or grooves.

"Taken alone, no theory as to the prevalence or otherwise of floating ice in the sea of the period during which the lower part of the chalk was deposited can be founded on this particular boulder. But it at all events supports the already existing theory, based on the character of the boulders and pebbles already described from the Cambridge Greensand. It has two characteristics of ice-borne erratics:—1. It is superficially like boulders recognised as having been transported from distant sources by ice, and subjected to the peculiar wear and tear incident to ice-action. 2. Its material is derived from a parent rock which can under no probable circumstances have existed, at the period of the chalk, within a very considerable distance of its recently discovered resting-place. We may therefore fairly, I think, accept it as evidence of the probability of the existence of floating ice in the sea of the chalk period[1]."

The rock is a very compact quartzfelsite of a dull purple colour. Examined with a hand-lens, small specks of quartz are seen to be scattered in the matrix, which exhibits some faint indications of a fluidal structure. The rock is singularly well preserved, the purple hue predominating up to the exterior surface, and marked indications of decomposition not extending inwards for more than a quarter of an inch. Considering that the rock has been lying for ages in a waterbearing stratum this strikes me as noteworthy. The rock reminds me in general aspect of specimens which I have seen in the volcanic breccias of the N. W. part of Charnwood Forest, especially of a compact purplish felsite (old rhyolite) from Timberwood Hill. I fully expected that there would be a close microscopic resemblance. This however is not the case. I have examined the chief varieties of the Charnwood fragments. They generally exhibit a "devitrified" matrix without definitely formed crystallites, or spherulites, and with but slight and rare traces of fluidal structure; in this matrix are small scattered crystals or crystalline grains of quartz and felspar. This generalization is founded upon eighteen slides cut from different specimens in my

[1] See a Paper by Messrs Sollas and Jukes-Browne, *Quart. Journ. Geol. Soc.* Vol. xxix. p. 113.

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collection. On placing the slide from the Cambridge Greensand erratic under the microscope, the difference from the Charnwood rhyolites is at once perceptible. Spherulites abound: the smaller aggregated in elongated clusters so as to give an irregular banded structure to the rock, which, indeed, on inspection with the lens, can just be seen on the smooth surface of the original fragment. The majority of these spherulites are about $\frac{1}{400}$ inch diameter, but they not seldom attain about $\frac{1}{100}$ inch, and sometimes even $\frac{1}{60}$. The structure is radial, but with some irregularity, so the 'black crosses' are not well defined. One or two darker concentric bands (more deeply stained with brown iron oxide) are often visible, and a great number are compound in structure, a larger outer ring enclosing two or three little spherules, like seeds in a husk. Beyond the outer clearer ring is generally an irregular (more darkly stained) zone of radiating crystallites, like the rays to a sun. The remaining part of the ground mass is occupied by interlacing clear acicular crystallites interspersed with an almost black residuum, exhibiting some tendency to radial aggregation. Scattered about this ground mass are small crystalline grains of rather clear quartz, with a few of decomposed felspar, and rather numerous granules and trichites of iron oxide—probably magnetite. In one place there appears to have been a small cavity partly filled by 'dirty' chalcedonic quartz.

Thus the microscopic structure of the rock differs very decidedly from any specimen which I have examined from Charnwood. It differs also from the old rhyolitic rocks of the Wrekin and of North Wales. Although it has a certain family likeness to all of these, enough to embolden one to suggest that it may have been derived from some volcanic mass, now lost to sight, which was ejected in the latest Precambrian epoch, I cannot venture to refer it to any locality known to me in Britain. I have however no doubt that the pebble described by Mr Watts (*Geol. Mag.* Dec. 2, Vol. VIII. p. 95) is from the same locality. Through his kindness I have again had the opportunity of examining this, and though the structure shewn by it is more distinctly banded and less definitely spherulitic than in Mr Fordham's specimen there are so many minute points of agreement, that I feel certain both have come from the same volcanic district.

(2) *On Critical or apparently neutral Equilibrium.* By J. LARMOR, M.A.

In the October Proceedings Mr Greenhill has translated the solution of some equations which occur in the analysis appended to my Note on Critical Equilibrium into Jacobi's elliptic function notation. I may observe, however, that the general solution of

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those equations was not the object I had in view, so I did not consider that I was required to pause in order to state it explicitly in the inverse notation.

On referring to the Note, it will be seen that I was discussing the oscillations of a rocking body whose centre of gravity is a very short distance on the unstable side of the critical position, and for which therefore c is very small: and the result is obtained by considering the small change produced in the critical period and the critical motion by the small amount of apparent instability.

But when c becomes at all large the character of the motion is quite altered: the flanking stable positions are now some distance from the centre of the oscillation, and we can no longer neglect higher powers of θ.

Thus Mr Greenhill's statement of the solution is not applicable to the problem when c is not small: and when c is small, it is unnecessarily complex. The reduction to a simple approximate form corresponding to that which I gave in the Note (in which the only modulus that occurs is sin 45°) would involve differentiation of the inverse functions with respect to the modulus: and I cannot see that such a process would lead to results simpler or more calculable than those which I obtained.

(3) *On the small free normal vibrations of a thin homogeneous and isotropic elastic shell, bounded by two confocal spheroids.* By W. J. Ibbetson, B.A.

[I.]

So far as I am aware no attempt has yet been made to solve a problem of this kind by the direct use of curvilinear coordinates, other than polars.

The present paper is limited to the case in which the shell vibrates in such a manner that its surfaces always remain spheroids confocal with their unstrained forms. This case is interesting because it is the only possible form of motion unaccompanied by shear in the substance of the shell, and because the results admit of numerical calculation.

It is obvious that throughout the motion the two systems of orthogonal surfaces (planes through the axis, and hyperboloids of one or two sheets, according as the shell is oblate or prolate) always remain the same.

The corresponding problem in polars is therefore to investigate the radial vibrations of a spherical shell of uniform thickness represented by an harmonic of zero order.

We shall see that the solution for the sphere can be readily deduced from the results obtained for the spheroid.

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Let the shell be bounded by the surfaces

$$\left.\begin{aligned} \frac{x^2+y^2}{a^2}+\frac{z^2}{c^2} &= 1 \\ \frac{x^2+y^2}{a^2+\epsilon}+\frac{z^2}{c^2+\epsilon} &= 1 \end{aligned}\right\},$$

where ϵ is a very small quantity compared with a^2 and c^2.

Take the orthogonal system of curvilinear coordinates ξ, η, ζ, where $+\xi$ and $-\eta$ are the roots of the quadratic

$$\left.\begin{aligned} \frac{x^2+y^2}{a^2+\lambda}+\frac{z^2}{c^2+\lambda} &= 1 \\ \text{and} \qquad \zeta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\}.$$

The unstrained surfaces of the shell are then given by

$$\xi = 0, \qquad \xi = \epsilon.$$

The limits of η are $+a^2$ and $+c^2$.

During the motion of any point of the shell η and ζ remain constant, while ξ is always a very small (positive or negative) quantity, and a function of the time only.

In the most general case of symmetrical motion

$$\xi = f(t, \eta).$$

It is easily shewn that

$$\left.\begin{aligned} x^2+y^2 &= \frac{(a^2+\xi)(a^2-\eta)}{a^2-c^2} \\ z^2 &= \frac{(c^2+\xi)(\eta-c^2)}{a^2-c^2} \end{aligned}\right\}.$$

Also, if ρ_1, ρ_2 be the principal radii of curvature of the strained shell at time t, the formulae in §§ 242, 181 of *Salmon's Conics* become

$$\left.\begin{aligned} \rho_1 &= \sqrt{\frac{(\xi+\eta)^3}{(\xi+a^2)(\xi+c^2)}} \\ \rho_2 &= \sqrt{\frac{(\xi+a^2)(\xi+\eta)}{\xi+c^2}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $h_1^2 = \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2$, and similarly for h_2, h_3.

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Then

$$\left.\begin{aligned} h_1 &= 2\sqrt{\frac{(\xi+a^2)(\xi+c^2)}{\xi+\eta}} \\ h_2 &= 2\sqrt{\frac{(a^2-\eta)(\eta-c^2)}{\xi+\eta}} \\ h_3 &= \sqrt{\frac{a^2-c^2}{(\xi+a^2)(a^2-\eta)}} \end{aligned}\right\}.$$

Let square brackets [ ] denote that the quantity enclosed is to have the value it assumes when $\xi = 0$.

Then if dS be the element of surface of the unstrained shell at either of the *two* points (η, ζ)

$$dS = \frac{d\eta \,.\, d\zeta}{[h_2 h_3]}$$

$$= \tfrac{1}{2}\sqrt{\frac{a^2\eta}{(a^2-c^2)(\eta-c^2)}} \,.\, d\eta \,.\, d\zeta.$$

Thus the integral, taken over the whole surface, of any function U of η only is, if the shell be oblate,

$$\frac{2\pi a}{\sqrt{a^2-c^2}}\int_{c^2}^{a^2} U \,.\, \sqrt{\frac{\eta}{\eta-c^2}} \,.\, d\eta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and if it be prolate,

$$\frac{2\pi a}{\sqrt{c^2-a^2}}\int_{a^2}^{c^2} U \,.\, \sqrt{\frac{\eta}{c^2-\eta}} \,.\, d\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The perpendicular from the centre on the tangent plane to the unstrained shell at (η, ζ) is $\frac{1}{2}[h_1]$. Hence if τ be the thickness at that point,

$$\tau = \frac{\epsilon}{2}\left[\frac{\partial h_1}{\partial \xi}\right]$$

$$= \frac{\epsilon}{2ac} \,.\, \frac{\eta(a^2+c^2) - a^2c^2}{\eta^{\frac{3}{2}}}.$$

The normal velocity at time t of the point (η, ζ) will be, approximately,

$$\frac{\dot{\xi}}{[h_1]} = \frac{\dot{\xi}}{2ac} \,.\, \sqrt{\eta}.$$

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Hence if ρ be the density of the shell, and T the kinetic energy of the motion, we get by writing $U=\frac{\rho}{2}\cdot\frac{\dot{\xi}^2}{4a^2c^2}\cdot\eta\cdot\tau$ in the integration formulæ (2) and (3)

$$T=\frac{\pi\rho\epsilon(a^4+2c^4)}{12\cdot a^2c^2}\cdot\dot{\xi}^2,$$

whether the shell be oblate or prolate.

Again, if v be the potential energy of the strain per unit of unstrained surface, we know that

$$v=\frac{q\tau^3}{24(1-\mu^2)}\left\{\left(\delta\frac{1}{\rho_1}\right)^2+\left(\delta\frac{1}{\rho_2}\right)^2+2\mu\delta\frac{1}{\rho_1}\cdot\delta\frac{1}{\rho_2}\right\},$$

where q is "Young's Modulus," μ the ratio of lateral contraction to longitudinal extension, and $\delta\frac{1}{\rho_1}$, $\delta\frac{1}{\rho_2}$ are the small increments of the principal curvatures due to the strain.

Hence, in our case,

$$v=\frac{q\tau^3\xi^2}{24(1-\mu^2)}\cdot\left\{\left[\frac{\partial}{\partial\xi}\frac{1}{\rho_1}\right]^2+\left[\frac{\partial}{\partial\xi}\frac{1}{\rho_2}\right]^2+2\mu\left[\frac{\partial}{\partial\xi}\frac{1}{\rho_1}\right]\left[\frac{\partial}{\partial\xi}\frac{1}{\rho_2}\right]\right\}.$$

Differentiating (1), and putting $\xi=0$,

$$v=\frac{q\tau^3\xi^2}{96(1-\mu^2)a^6c^3\eta^5}\cdot\{\eta^2[\eta(a^2-c^2)-a^2c^2]^2$$
$$+a^4[\eta(a^2+c^2)-3a^2c^2]^2+2\mu a^2\eta[a^4(\eta-2c^2)^2-c^4(\eta-a^2)^2]\}.$$

Hence if W_1, W_2 be the integrals of the potential energy for the prolate and oblate shells, we get by putting $U=v$ in (2) and (3) successively,

$$W_1=\frac{\pi q\epsilon^3\xi^2}{384(1-\mu^2)a^6c^5\sqrt{a^2-c^2}}\int_{c^2}^{a^2}\{\eta^2[\eta(a^2-c^2)-a^2c^2]^2$$
$$+a^4[\eta(a^2+c^2)-3a^2c^2]^2+2\mu a^2\eta[a^4(\eta-2c^2)^2-c^4(\eta-a^2)^2]\}$$
$$\{\eta(a^2+c^2)-a^2c^2\}^3\frac{d\eta}{\eta^9\sqrt{\eta-c^2}}.$$

$$W_2=\frac{\pi q\epsilon^3\xi^2}{384(1-\mu^2)a^6c^5\sqrt{c^2-a^2}}\int_{a^2}^{c^2}\{\eta^2[\eta(a^2-c^2)-a^2c^2]^2$$
$$+a^4[\eta(a^2+c^2)-3a^2c^2]^2+2\mu a^2\eta[a^4(\eta-2c^2)^2-c^4(\eta-a^2)^2]\}$$
$$\{\eta(a^2+c^2)-a^2c^2\}\frac{d\eta}{\eta^9\sqrt{c^2-\eta}}.$$

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It will be convenient at this stage to make the a and c of the prolate shell respectively equal to the c and a of the oblate shell. These symbols will then denote in all the formulæ the major (a) and minor (c) semiaxes of the meridional section common to the two shells.

Thus, interchanging a and c in the prolate formulæ, and writing

$$\alpha = \frac{c}{a}, \quad \beta^2 = \frac{\epsilon}{a^2},$$

we have for the oblate shell

$$\left.\begin{aligned} T_1 &= \frac{\pi}{12} \cdot \frac{\rho a \beta^2 (1 + 2\alpha^4)}{\alpha^3} \cdot \dot{\xi}^2 \\ W_1 &= \frac{\pi}{12} \cdot \frac{q \beta^6 A_1}{16 (1 - \mu^2) a \alpha^{12}} \cdot \xi^2 \end{aligned}\right\},$$

and for the prolate shell

$$\left.\begin{aligned} T_2 &= \frac{\pi}{12} \cdot \frac{\rho a \beta^2 (2 + \alpha^4)}{\alpha^2} \cdot \dot{\xi}^2 \\ W_2 &= \frac{\pi}{12} \cdot \frac{q \beta^6 A_2}{16 (1 - \mu^2) a \alpha^8} \cdot \xi^2 \end{aligned}\right\},$$

where A_1 and A_2 are functions of α only, which we may write in the form

$$A_1 = \frac{1}{\sqrt{1 - \alpha^2}} \int_0^{\cos^{-1}\alpha} \{\alpha^4 (\sin^2\theta - \alpha^2)^2 + \cos^4\theta (1 + \alpha^2 - 3\cos^2\theta)^2 + 2\mu\alpha^2 \cos^2\theta [\cos^2 2\theta - (\alpha^2 - \cos^2\theta)^2]\} (\sin^2\theta + \alpha^2)^3 \cdot \cos^2\theta \cdot d\theta.$$

$$A_2 = \frac{1}{\sqrt{1 - \alpha^2}} \int_0^{\cos^{-1}\alpha} \{\cos^4\theta (\cos^2\theta + \alpha^2 \sin^2\theta)^2 + \alpha^4 [(1 + \alpha^2)\cos^2\theta - 3\alpha^2]^2 + 2\mu\alpha^2 \cos^2\theta [\alpha^4 (1 + \sin^2\theta)^2 - (\cos^2\theta - \alpha^2)^2]\} (\cos^2\theta - \alpha^2 \sin^2\theta)^3 \cdot \frac{d\theta}{\cos^{17}\theta}.$$

These integrations are easily performed, term by term, though the calculation of the numerical coefficients is tedious.

The results may be written

$$A_1 = h_0 + h_1\alpha^2 + \ldots + h_7\alpha^{14} + (k_0 + k_1\alpha^2 + \ldots + k_7\alpha^{14}) \frac{\cos^{-1}\alpha}{\sqrt{1 - \alpha^2}},$$

$$A_2 = m_0 + m_1\alpha^2 + \ldots + m_7\alpha^{14} + (n_0 + n_1\alpha^2 + \ldots + n_7\alpha^{14}) \frac{\log\left(\frac{1 + \sqrt{1 - \alpha^2}}{\alpha}\right)}{\sqrt{1 - \alpha^2}}.$$

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The equations of free vibration of the two shells become

$$\left.\begin{aligned} \ddot{\xi} + \frac{q}{16\rho(1-\mu^2)} \cdot \frac{\beta^4 A_1}{(1+2\alpha^4)\alpha^2 a^2} \cdot \xi = 0 \\ \ddot{\xi} + \frac{q}{16\rho(1-\mu^2)} \cdot \frac{\beta^4 A_2}{(2+\alpha^4)\alpha^6 a^2} \cdot \xi = 0 \end{aligned}\right\},$$

and if t_1, t_2 be the periodic times,

$$\left.\begin{aligned} t_1 = \frac{8\pi a}{\beta^2} \sqrt{\frac{\rho(1-\mu^2)}{q} \cdot \frac{\alpha^9(1+2\alpha^4)}{A_1}} \\ t_2 = \frac{8\pi a}{\beta^2} \sqrt{\frac{\rho(1-\mu^2)}{q} \cdot \frac{\alpha^6(2+\alpha^4)}{A_2}} \end{aligned}\right\}.$$

Thus the times of vibration of similar shells of the same kind are in the simple ratio of their linear dimensions.

As a verification of these results we may deduce from either the case of a thin spherical shell of radius a and uniform thickness τ, performing small radial vibrations so as always to retain the spherical form.

We are to put

$$\alpha = 1, \quad \beta^2 = \frac{2\tau}{a}.$$

Now A_1 and A_2 are both of the form

$$\frac{1}{\sqrt{1-\alpha^2}} \int_0^{\cos^{-1}\alpha} F(\alpha, \theta)\, d\theta,$$

and when $\alpha = 1$, this becomes by evaluation $F(1, 0)$, which $= 2(1+\mu)$ in both cases.

Hence our formulæ reduce to

$$t_1 = t_2 = \frac{4\pi a^2}{\tau} \sqrt{\frac{3}{2} \cdot \frac{\rho(1-\mu)}{q}}.$$

Now, with our previous notation, if $a+u$ be the radius of the spherical shell at time t,

$$T = \tfrac{1}{2}\rho \cdot 4\pi a^2 \cdot \tau \cdot \dot{u}^2,$$

and

$$v = \frac{q\tau^3}{24(1-\mu^2)} \cdot 2(1+\mu) \left[\delta \frac{1}{a+u}\right]_{u=0}^2;$$

$$\therefore \quad W = 4\pi a^2 \cdot \frac{q\tau^3}{24(1-\mu^2)} \cdot 2(1+\mu) \frac{u^2}{a^4},$$

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and the equation of motion is

$$\ddot{u} + \frac{q}{6\rho(1-\mu)} \cdot \frac{\tau^2}{a^4} \cdot u = 0.$$

Hence the periodic time is

$$\frac{2\pi a^2}{\tau} \cdot \sqrt{\frac{6\rho(1-\mu)}{q}},$$

which is the result already obtained.

(4) *On the curves of constant intensity of homogeneous polarized light seen in a uniaxal crystal cut at right angles to the optic axis.* By C. SPURGE, B.A.

Introduction.

Sir George Airy has shown in his Tract on the Undulatory Theory that if a plate of Iceland spar bounded by planes perpendicular to the axis of the crystal be placed between a polarizing and analyzing plate the brightness of any point of the image formed after passing the analyzer is given by the formula

$$a^2\left\{\cos^2\alpha - \sin(2\psi - 2\alpha)\sin 2\psi \sin^2\frac{\pi I}{\lambda}\right\}$$

where I is proportional to the square of the radius vector from the centre of the image to the point considered and ψ is the other polar co-ordinate of the point considered, viz. the angle the radius vector makes with a fixed line. α is the angle between the plane of polarization at the analyzing plate and the plane of first polarization. If $\alpha = 0$ or $\frac{\pi}{2}$ we have for the intensity

$$a^2\left\{1 - \sin^2 2\psi \sin^2\frac{\pi I}{\lambda}\right\}$$

and $$a^2 \sin^2 2\psi \sin^2\frac{\pi I}{\lambda}$$ respectively.

Thus in either case the curves of equal intensity are given by an equation of the form

$$\text{Constant} = \sin^2 2\psi \sin^2\frac{\pi I}{\lambda},$$

or $$k^2 = \sin^2 2\psi \sin^2 r^2$$

where k^2 is put for the constant

and $$r^2 = \frac{\pi I}{\lambda},$$

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so that r may still be treated as a radius vector, the effect of putting r^2 for $\frac{\pi I}{\lambda}$ being merely to alter the size of the image in a certain ratio.

The object of the present paper is first to obtain the mathematical properties of these curves and secondly by means of actual calculation to trace with accuracy the curves and thus obtain their exact form.

Some of the properties have been given in a paper by Mr R. T. Glazebrook, *Proc. Camb. Phil. Soc.*, Vol. IV. Part VI.

General Properties of the Curve.

$$k^2 = \sin^2 2\theta \sin^2 r^2$$

or

$$k = \pm \sin 2\theta \sin r^2.$$

The equation $k = \pm \sin 2\theta \sin r^2$ is in Polar Co-ordinates. When in the investigation which follows Rectangular Co-ordinates are mentioned it is to be understood that the origin is the same as for Polar Co-ordinates, that the Prime Vector is taken as the axis of x and a line perpendicular to it through the origin as the axis of y.

PROPOSITION I. *The curve is symmetrical in the four quadrants.*

This is obvious from the form of the equation.

PROPOSITION II. *The curve (for a given value of k) consists of an infinite number of closed curves.*

When $k = 0$ the parts of the closed curves combine so that the system of curves may be regarded as a number of (double) quadrants of circles and (double) straight lines Ox, Oy. See Glazebrook *loc. cit.*

These curves we shall term ovals, and for the sake of distinction we shall give them the names first, second,...nth ovals of intensity k. Thus the first oval of intensity k is given by

$$r^2 = \rho^2 \text{ and } r^2 = \pi - \rho^2.$$

The second by

$$r^2 = \pi + \rho^2 \text{ and } r^2 = 2\pi - \rho^2,$$

and the nth oval will be given by

$$r^2 = \overline{n-1}\,\pi + \rho^2 \text{ and } r^2 = n\pi - \rho^2 \quad \ldots\ldots\ldots\ldots (\alpha),$$

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when we now take ρ^2 to be the least positive value of r^2 which satisfies

$$\sin r^2 = \pm \frac{k}{\sin 2\theta}.$$

PROPOSITION III. *Form of the curve near the origin.*

The form of the curve near the origin is a rectangular hyperbola.

The general equation of the curve is

$$k = \pm \sin 2\theta \sin r^2.$$

For points near the origin r is very small so that for $\sin r^2$ we may put r^2 neglecting quantities of the sixth order. Thus the equation of the curve near the origin is

$$k = \pm \sin 2\theta \,.\, r^2$$

or transforming to Cartesian co-ordinates

$$\frac{k}{2} = \pm xy.$$

The equation of the rectangular hyperbola referred to the axes of x and y as asymptotes.

PROPOSITION IV. *Form of the curve at points remote from the origin.*

The ovals become arcs of circles except towards their extremities which are rounded off (provided the ovals are not too near the point curves).

The equation to the curve is

$$\sin r^2 = \pm \frac{k}{\sin 2\theta}.$$

As in Prop. II. let ρ^2 be the least positive value of r^2 which satisfies this equation.

Then by equations α the general value of r^2 is

$$r^2 = n\pi \pm \rho^2$$

where ρ^2 is less than $\frac{\pi}{2}$.

Thus $$r = \sqrt{n\pi \pm \rho^2} = \sqrt{n\pi} \pm \frac{\rho^2}{2\sqrt{n\pi}} \pm \&c. \ldots\ldots\ldots\ldots\ldots\ldots (D).$$

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Also ρ^2 is given by $\sin \rho^2 = \pm \frac{k}{\sin 2\theta}$.

Near the middle of the curve, *i.e.* when $\theta = \frac{\pi}{4}$, $\sin 2\theta$ which is then at its maximum will vary slowly; thus $\sin \rho^2$ will vary slowly. Now ρ^2 near the middle of the curve is not nearly a multiple of $\frac{\pi}{2}$ because the curve is not near the point curve. Therefore the value of ρ^2 also varies slowly. Again from the formula (D) that part of r^2 which contains ρ^2 is divided by $2\sqrt{n\pi}$. Hence for ovals of a high order the term $\frac{\rho^2}{2\sqrt{n\pi}}$ will be almost invariable first because ρ^2 does not vary much, and secondly because its variation is divided by a considerable factor $2\sqrt{n\pi}$. Thus near the middle of the curves at a distance from the origin the ovals become circular arcs,

for $$r = \sqrt{n\pi} \pm \frac{\rho^2}{2\sqrt{n\pi}}$$

is almost invariable.

The contrary will be the case at the ends of the curve for here ρ^2 is equal to a multiple of $\frac{\pi}{2}$ (odd multiple that is). Thus from the slow variation of $\sin \rho^2$ we cannot infer a correspondingly slow variation of ρ^2. Consequently the ends of the curve will be rounded off.

PROPOSITION V. *All ovals of the same intensity touch two straight lines passing through the origin, and the ovals are entirely comprised between these straight lines.*

See Glazebrook, loc. cit.

Cor. This proposition may be stated in various ways, *e.g.* all ovals of the same intensity subtend the same angle at the origin.

PROPOSITION VI. *If tangents be drawn from the origin to all* n$^{\text{th}}$ *ovals their points of contact lie on a circle.*

In the preceding proposition we have just shewn that the tangents from the origin to the ovals of intensity k are the lines

$$\theta = \tfrac{1}{2} \sin^{-1} k,$$
$$\theta = \tfrac{1}{2}\pi - \tfrac{1}{2} \sin^{-1} k.$$

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The general equation being

$$\pm \frac{k}{\sin 2\theta} = \sin r^2,$$

we have for the radii of the points of contact the equation

$$\sin r^2 = \pm \frac{k}{k} = \pm 1,$$

substituting k for $\sin 2\theta$.

If the point of contact is on the nth oval we must take the solution

$$r^2 = \overline{2n-1}\,\frac{\pi}{2}.$$

Thus the distance of the point of contact of the tangents to the nth oval of intensity k is

$$r = \sqrt{(2n-1)\,\frac{\pi}{2}}.$$

Now this radius is independent of k but contains n. Therefore the tangents to all nth ovals have their points of contact on a circle centre the origin and radius $\sqrt{\overline{2n-1}\,\frac{\pi}{2}}$.

PROPOSITION VII. *All ovals of the same intensity have the same area.*

See Glazebrook, loc. cit.

PROPOSITION VIII. *Let the tangents be drawn from the origin to ovals of intensity* k. *The area contained by the tangents and the two parts of any two consecutive ovals that they intercept will be always the same.*

Consider the area between the parts intercepted by the tangents to the nth and $n+1$th ovals of intensity k.

In Fig. 1, PP' is the nth oval, QQ' the $n+1$th oval and $PP'Q'Q$ is the element bounded by the curves and lines through the origin. OEE' OFF' the common tangents, see Prop. V. Just as in the last proposition

$$\text{Area } EE'F'F = \tfrac{1}{2}\int (OQ^2 - OP^2)\, d\theta.$$

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But by equation α, Prop. II.,

$$OQ^2 = n\pi + \rho^2$$
$$OP^2 = n\pi - \rho^2.$$

$$\therefore \text{ Area } EE'QF'FPE = \tfrac{1}{2}\int 2\rho^2 d\theta = \int \rho^2 d\theta.$$

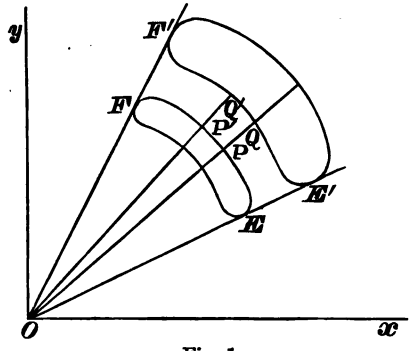

Fig. 1.

The limits are $\theta = \frac{1}{2}\sin^{-1} k$ to $\theta = \frac{\pi}{2} - \frac{1}{2}\sin^{-1} k$, and as in last proposition ρ^2 is independent of n.

Therefore the area is independent of n. Thus the area between the first and second ovals will be the same as between the nth and $n+1$th ovals.

Cor. I. Let T_k denote the area between tangents and ovals of k^p intensity.

By the preceding

$$T_k = \int_{\frac{1}{2}\sin^{-1}k}^{\frac{\pi}{2}-\frac{1}{2}\sin^{-1}k} \rho^2 d\theta.$$

By Prop. VII. if A_k denote the area of an oval of intensity k

$$A_k = \frac{\pi}{2}\left[\frac{\pi}{2} - \sin^{-1} k\right] - \int_{\frac{1}{2}\sin^{-1}k}^{\frac{\pi}{2}-\frac{1}{2}\sin^{-1}k} \rho^2 d\theta.$$

Therefore $$T_k + A_k = \frac{\pi}{2}\left[\frac{\pi}{2} - \sin^{-1} k\right].$$

Cor. II. A Prop. similar to Prop. VIII. holds if instead of areas between consecutive ovals the areas between ovals whose difference of order is constant be taken.

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Thus for example the areas between tangents and n^{p} and $n+2^{ds}$ ovals are the same whatever be the value of n.

PROPOSITION IX. *Each oval of the* nth *order is bisected by the circle which is the locus of the points of contact of tangents drawn to them from the origin.* (See Prop. VI.)

The proof of this involves no difficulty.

PROPOSITION X. *Let tangents be drawn from the origin to the ovals. The area contained by the tangents and the parts of any two consecutive ovals they intercept will be bisected by the quadrantal arc of intensity* $k = 0$ *which passes between the two consecutive ovals.*

This also admits of a very simple proof.

PROPOSITION XI. *Generally if two consecutive lines be drawn through the origin to meet two ovals each of the* nth *order and of* lth *and* mth *intensities the area of the element formed by the ovals and the parts of the consecutive lines intercepted by them is independent of* n.

PROPOSITION XII. *If a vector be drawn in any direction through the origin meeting any* nth *ovals in points* R_n, R_n',

$$OR_n^2 + OR_n'^2 = \textit{constant}.$$

Let the line OR_nR_n' meet an nth oval of intensity k.

By Prop. II., equations α,

$$OR_n^2 = \overline{n-1}\,\pi + \rho^2,$$
$$OR_n'^2 = n\pi - \rho^2,$$

where ρ^2 (only) depends on k.

Therefore $$OR_n^2 + OR_n'^2 = \overline{2n-1}\,\pi.$$

Since the right hand does not contain k the Prop. holds for all ovals of the nth order.

Also the constant is independent of θ the direction of the line, and the proposition holds for all directions of the line.

Cor. If $OR_n'R_n$ be produced to meet the $n+1$th oval of intensity k in R_{n+1}, we have

$$OR_{n+1}^2 = n\pi + \rho^2.$$

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But $OR_n'^2 = n\pi - \rho^2$.

Therefore $OR_{n+1}^2 + OR_n'^2 = 2n\pi$

= a constant independent of the intensity k.

PROPOSITION XIII. *If a vector be drawn through the origin meeting any* k*th intensity ovals in* R_k, R_k',

$$OR_k'^2 - OR_k^2 = \textit{constant}.$$

This can be proved in a manner similar to Prop. XII.

PROPOSITION XIV. *If a circle be described from the origin* O *as centre meeting the* n*th ovals in points* A_1, A_2 ... *and* OA_1, OA_2 ... *be produced to meet the* n*th ovals again in* $A_1'A_2'$ ... $A_1'A_2'$ ... *will lie on another circle.*

For by Prop. XII.,

$$OA_1^2 + OA_1'^2 = \overline{2n-1}\,\pi.$$

But $OA_1^2 =$ a constant $= a^2$ say;

$$\therefore\ OA_1'^2 = \overline{2n-1}\,\pi - a^2.$$

Thus $A_1'A_2'$ ... lie on an nth oval,

OA_1', OA_2' ... &c. are all constant.

Therefore A_1', A_2' ... lie on a circle-centre the origin and of radius $\sqrt{(2n-1\,\pi - a^2)}$.

Cor. I. If OA_1', OA_2' ... be produced to meet the next set of $n+1$th ovals in A_1'', A_2'' ... $A_1''A_2''$ lie on a circle.

Cor. II. If production be continued the radius of the next circle is

$$\sqrt{(2n+1\,\pi - \pi - a^2)},$$

or $\sqrt{2n\pi - a^2}$.

Thus we see the squares of the radii of alternate circles increase in Arithmetical Progression.

PROPOSITION XV. *The moments of inertia of the ovals of same intensity considered of uniform density form a series of terms in Arithmetical Progression, the moments of inertia being supposed taken about an axis through the origin perpendicular to the plane of the curves.*

This follows by the aid of Prop. XIII.

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PROPOSITION XVI. *The curvature of the ovals at points where tangents from the origin touch them varies directly as the cube of the distance of the points from the origin.*

Let OQ (Fig. 2) be the tangent to the nth oval, OP a vector consecutive to OA.

Let $$POQ = d\theta.$$

We have $\sin 2\theta = k$ for the line OQ,

$$OQ^2 = r^2 = \overline{2n-1}\,\frac{\pi}{2}.$$

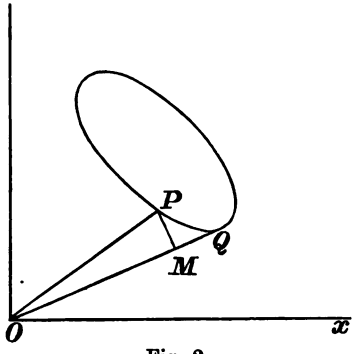

Fig. 2.

Now at the outset we may observe that it follows from the geometry (i.e. since OQ is a tangent) that if dr the increment of OQ be of the first order of infinitesimals $d\theta$ will be of the second order of infinitesimals. Hence we may expect the relation between $d\theta$, dr to assume the form

$$\overline{dr}|^2 = A\,d\theta.$$

Consequently in the expansion that follows we keep the terms dr up to the second order, the terms $d\theta$ to the first order of infinitesimals only.

The general equation of the curve is

$$\sin r^2 = \pm\frac{k}{\sin 2\theta}.$$

To find the relation between dr and $d\theta$ of point Q we substitute for r^2 $(r+dr)^2$, for θ $\theta + d\theta$.

Thus we obtain

$$\sin (r+dr)^2 = \pm\frac{k}{\sin 2(\theta+d\theta)},$$

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or
$$\sin r^2 \cos(2rdr + \ldots) + \cos r^2 \sin(\ldots)$$
$$= \pm \frac{k}{\sin 2\theta} \mp \frac{2k \cos 2\theta \, d\theta}{\sin^2 2\theta},$$
expanding this side by simple differentiation.

Now since
$$r^2 = \overline{2n-1}\,\frac{\pi}{2},$$
$$\cos r^2 = 0 \text{ and } \sin r^2 = \pm \frac{k}{\sin 2\theta}; \quad \sin 2\theta = k$$
$$= \pm 1;$$
$$\therefore \pm \left(1 - \frac{4r^2 \overline{dr}^2}{2}\right) \frac{k}{\sin 2\theta} = \pm \frac{k}{\sin 2\theta} \mp \frac{2k\sqrt{1-k^2}}{k^2} d\theta.$$

Whence cancelling terms we have as the relation between dr and $d\theta$
$$\frac{4r^2 \overline{dr}^2}{2} = \frac{2\sqrt{1-k^2}}{k^2} d\theta;$$
$$\therefore \frac{dr^2}{rd\theta} = \frac{\sqrt{1-k^2}}{k} \frac{1}{r^3}.$$

But if in figure PM be perpendicular to OQ
$$PM = rd\theta,$$
$$MQ = dr,$$
and ρ the radius of curvature $= \dfrac{dr^2}{2rd\theta}$.

Thus
$$2\rho = \frac{\sqrt{1-k^2}}{k} \frac{1}{r^3}.$$

Therefore the curvature varies directly as the cube of the distance r of the points of contact from the origin.

Also we see if $k = 0$, $\rho = \infty$, this gives us the flattened parts of curves for $k = 0$, and serves to verify the formula.

If $k = 1$, $\rho = 0$ gives the point circles to which the ovals diminish when $k = 1$.

PROPOSITION XVII. *If tangents be drawn from the origin to all ovals of intensity* k *the centres of circles of curvature at points of contact lie on the unicursal quartic*
$$yx^3 = \frac{\sqrt{1-k^2}}{2k}.$$

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when the axis of x *is the tangent to all the ovals and the origin is at the usual place.*

This follows from the value of ρ found in Prop. XVI.

PROPOSITION XVIII. *If a straight line be drawn through the origin to meet ovals of intensity* k, *the area of the triangle formed by the vectors to points of section and corresponding subnormals is constant.*

The equation to the curve is

$$\sin r^2 = \pm \frac{k}{\sin 2\theta}.$$

Therefore
$$r^2 = \pm \sin^{-1} \frac{k}{\sin 2\theta} + n\pi,$$

n being an integer.

Therefore by differentiation

$$\frac{rdr}{d\theta} = \pm \text{ function of } \theta, k.$$

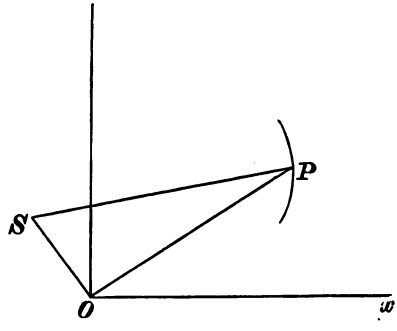

Fig. 3.

Let OP (fig. 3) be the line through origin meeting the oval of intensity k in P, and PS the normal, OS the polar subnormal.

Then, if $OP = r$, $Pox = \theta$,

$$\cot OPS = \frac{rd\theta}{dr},$$

$$\therefore \frac{rdr}{d\theta} = r^2 \tan OPS$$

$$= r \cdot OS,$$

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But $\frac{rdr}{d\theta}$ = function of θ, k, and is the same for all points on the line OP and on an oval of intensity k.

$$\therefore\ r \,.\, OS = \text{constant}.$$

Hence the area of triangle OSP is constant.

Therefore we have

PROPOSITION XIX. *All normals to ovals of intensity* k *drawn at points on a straight line through the origin touch a rectangular hyperbola.*

Cor. The tendency of normals at a distance is to become coincident with OP, for the hyperbolic curve has its asymptote along OP.

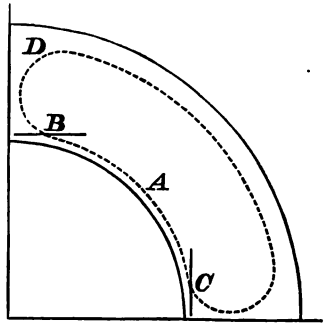
Thus again we see the curves are circular at distance from origin.

PROPOSITION XX. *The form of the curves near the point curves is elliptic, and the centre of such ellipses is very nearly at the point curve.*

This follows from the equation without difficulty.

PROPOSITION XXI. *Ovals of the second and higher orders have points of inflexion if the ovals are sufficiently near to the circle of intensity* k = 0.

Consider an oval of the second order which is near to a circle of intensity $k = 0$.



The portion of the curve near A being very nearly of a circular

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form, for its form can only slightly differ from that of an oval of intensity $k = 0$.

Thus the curve near A has its concavity *inwards*.

In order that the curve may turn so as to become closed it is clear that the concavity of the end BD must be *outwards*.

Thus a point of inflexion must exist at B somewhere between A and D.

Similarly there is by the symmetry of the curves a point of inflexion at C.

Thus on a curve near to an oval of intensity 0 there are two points of inflexion.

The equation for finding these points is not difficult to obtain. But it will be found to be so exceedingly complex that it is practically useless, and on this account it is not given.

(5) *Tables of the number of numbers not greater than a given number and prime to it, and of the number and sum of the divisors of a number, with the corresponding inverse tables, up to* 3000. *By* J. W. L. GLAISHER, M.A., F.R.S.

[*Abstract.*]

Denoting by $\phi(n)$ the number of numbers not greater than n and prime to it, by $\nu(n)$ the number of divisors of n, and by $\sigma(n)$ the sum of the divisors of n, unity and n itself being included, the tables contained in the present paper are as follows:

Table I. The complete resolution of n into factors and the values of $\phi(n)$, $\nu(n)$, and $\sigma(n)$ for all values of n up to $n = 3000$.

Table II. The values of n corresponding to $\phi(n)$ as argument.

Table III. The values of n corresponding to $\nu(n)$ as argument.

Table IV. The values of n corresponding to $\sigma(n)$ as argument.

Tables II.–IV. are inverse to Table I. and extend also to $n = 3000$.

An introduction containing a collection of formulæ relating to the functions $\phi(n)$, $\nu(n)$ and $\sigma(n)$ is prefixed to the paper.

The paper is in course of publication in the *Transactions* of the Society.

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February 11, 1884.

Mr GLAISHER, PRESIDENT, IN THE CHAIR.

The following communications were made:

(1) *On the constitution of the Cell-wall and Middle Lamella.* By Walter GARDINER, B.A.

IF we accept the view of Schmitz[1] and Strasburger[2] as to the first formation of the cell-wall, we must regard the cell-plate which appears as a delicate septum between the two dividing nuclei, and at the equator of the achromatin spindle, as consisting of a number of microsomes imbedded in a protoplasmic matrix[3].

The cell-wall which is subsequently formed, is the expression of a chemical change having taken place in the structure, in consequence of which cellulose appears as one of the principal bye-products: that is to say: that from the breaking down of a complex proteid, a carbohydrate has been produced.

The succeeding thickening of this primitive cellulose membrane[4], is brought about in a manner quite similar to that which attended its first formation, viz. by the repeated apposition upon it, of fresh sheets of microsome-laden protoplasm, which are successively deposited and converted into layers of cellulose, except that in the case of the secondary thickening, the microsomes are deposited by the general parietal protoplasm, and are not conveyed in the special fibrils of the achromatin spindle, which disappear with the formation of the primary wall.

Simultaneously with this thickening other phenomena usually occur. In the first place owing both to chemical alteration, and to pressure and tension, a distinct median layer becomes differentiated in the hitherto homogeneous wall which separates the contents of adjacent cells, so that the wall appears to be no longer common, but on the contrary, each cell-wall appears to be surrounded with its own cell-membrane, and, at the junction of the two is the well defined layer in question, which is usually known as the middle lamella.

[1] Schmitz, *Sitzber. niederrhein. Ges. in Bonn*, Dec. 6th, 1880.

[2] Strasburger, *Bau und Wachsthum*, p. 173.

[3] Were the idea borne out by observed facts, it would be simpler for the cell-wall to consist simply of aggregated microsomes, which gradually coalesce into a firm membrane, the microsomes being regarded as formed proteid substance, and hence midway between protoplasm on the one hand and cellulose on the other.

[4] In certain instances it would appear that the primitive membrane may not consist solely of cellulose, e.g. the seed of *Ardisia* where starch is present. The statement here refers only to typical walls.

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In the substance of the walls themselves also, changes both of a chemical and physical nature supervene, in consequence of which, the typically cellulose membrane becomes lignified, cuticularised, corky or mucilaginous as the case may be.

As we follow the history of the development of botanical histology and microchemistry, we find that in each of the above-mentioned kinds of cell-walls, certain substances were discovered, to the properties of which, the peculiar characteristics of the wall were due. Thus from the cuticle first described by Brogniart[1], Frémy[2] isolated a substance, to which he appropriated the name Cutin, and for which he gave the percentage composition, C. 73·66, H. 11·37, O. 14·97. In the same way lignified walls are usually allowed to owe their properties to the presence of lignin[3]. In corky walls, Chevreul[4] found a substance, which he called Suberin, which according to Doepping[5] gave ceric acid (impure suberic acid) after treatment with nitric acid, and was expelled from a corky tissue in yellow drops on treatment with boiling potash[6]. As to mucilaginous walls, the presence in them of mucilage or gum was long known. Kützing[7] (by boiling) isolated from the cells of certain algae, e.g. *Sphaerococcus crispus*, a mucilage to which he gave the name Phytogelin, and also recognized that gum-tragaganth consisted of a mass of swollen cell-walls, the cells of which often contained starch grains[8]. More recently also Frémy[9] described as pectose, a body allied to mucilage which is found in many unripe fruits and Berg[10] in investigating *Cetraria islandica* separated from it Lichen-starch and Lichenin.

At a very early period[11] it was recognized that in cells which had become lignified, cuticularised, mucilaginous or corky, an unaltered cellulose portion still remained, and the foreign matters (lignin, cutin, &c.) which prevented the visible occurrence of the cellulose reactions, were regarded as infiltrated substances. Thus Von Mohl found that in the case of epidermal and corky cells, a lengthy treatment with potash, dissolved out the foreign matter, and the remaining cell-wall now gave a blue coloured with iodine.

[1] Brogniart, *Ann. de Sci. Nat.* Ser. I. t. xxi. p. 427 (1830).
[2] Frémy, *Ann. de Sci. Nat.* Ser. IV. t. xii. (1859).
[3] Payen, *Mém. sur les développements des végétaux* (1844).
[4] Chevreul, Sur le moyen à analyser plusieurs matières végétales, &c. *Ann. de Chemie*, t. 96 (1815).
[5] Doepping, *Ann. Chem. u. Pharm. von Liebeg u. Wöhler*, Bd. 45, p. 286 (1843).
[6] Von Höhnel, Ueber Kork, &c. *Sitzber. d. k. Akad. in Wien*, LXXVI. (1877).
[7] Kützing, *Phycologia generalis.*
[8] Kützing, *Grund. d. phil. Bot.* 203, 204.
[9] Frémy, *Ann. de Sci. Nat.* Ser. 6, t. XIII. (1882).
[10] Berg, Zur Kenntniss des in Cetraria islandica &c. Diss. Dorpat, 1872.
[11] Scharcht, *The Microscope*, English translation, p. 69. Von Mohl, *Vegetable Cell*, English translation, p. 28.

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Woody and sclerenchymatous tissue also, after being treated with nitric acid and potash, reacted in the usual manner towards cellulose tests. Similarly as regards the cuticle, Hofmeister[1] found that after three weeks maceration in potash, the insoluble remaining skeleton became distinctly blue with a solution of iodine. Thus in every instance a cellulose framework apparently remains. The lignin, cutin, suberin and the like, must however be regarded, not as infiltrated substances, to be placed in the same category with such bodies as silica[2], or iron[3], but rather as portions of the cellulose which have experienced chemical change. In the present state of the science we know but little as to how these changes occur, but it is a matter of interest to observe that they can take place only in the living plant, although as it appears, in the conifers at least, the cells may have lost their cell-contents[4].

Besides these fairly well defined modifications of cellulose which accompany lignification, cuticularisation, or the formation of cork, we have yet to consider other forms of cell-wall, which may be conveniently separated from the foregoing. Sometime after Payen's[5] discovery that cell-walls turned blue when treated with iodine and sulphuric acid, Scharcht[6] showed that the walls of certain fungi, even after treatment with potash, did not give the reactions of cellulose, but with iodine and sulphuric acid were merely stained yellow. In consequence of this observation, the walls were regarded as consisting of a definite form of cellulose, which was known as the fungin of Braconnot[7], or the fungus cellulose of de Bary[8]. Richter[9] however is of opinion that a definite fungus cellulose does not exist, but that the walls in reality consist of ordinary cellulose together with a body, which in certain fungi, e.g. *Daedalea*, he makes out to be suberin[10]. He found that when fungus-tissue was treated for some time—in certain cases, several weeks—with solution of potash, washed with weak acid, and mounted in chlor. zinc. iod., the customary blue colouration was obtained. However, as Dragendorff[11] points out, one cannot with certainty make deductions with regard to results produced after such lengthy action of potash, and consequently, one must not regard them as quite decisive. Dragendorff[12]

[1] Hofmeister, *Pflanzenzelle*, p. 257.
[2] Von Mohl, *Bot. Zeit.* 1861.
[3] Weiss u. Wiesner, *Sitzber. d. k. Akad. in Wien*, XL. 1860.
[4] Strasburger, *Der Zellhaüte*, p. 199.
[5] Payen, *loc. cit.* 1844.
[6] Scharcht, *Die Pflanzenzelle*, 1852.
[7] Braconnot. See Fremy, *l. c.*
[8] De Bary, Morphologie der Pilze, Flechten u. Myxomyceten. (Hofmeister, *Handb. d. phys. Bot.* II. p. 7 u. ff.)
[9] Richter, *Sitzber. d. k. Akad. in Wien*, LXXXIII. 1881.
[10] In the mushrooms he believes that a substance of a proteid nature is present together with the cellulose.
[11] Dragendorff, *Plant Analysis*, English translation, 1884, p. 257.
[12] Dragendorff, *loc. cit.* p. 255.

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however is himself inclined to believe, that the hardening substances of many of the fungi of a woody character is identical with suberin. Frémy's[1] experiments, as I shall state later on, point to the occurrence of a substance which he calls metacellulose, in the tissue of lichens and fungi. Lastly, as before mentioned, Berg[2] succeeded in isolating lichenin and lichen starch, the latter being thus designated since it turns blue with iodine. We thus have evidence that in fungi as well as in woody and corky tissue the cell-walls consist of a framework of cellulose, and that other substances such as suberin, metacellulose, lichenin and the so-called lichen-starch may also be present.

Returning now to the consideration of ordinary vegetable cells we find that Von Mohl[3] made the observation that certain cell-walls existed, e.g. endosperm of *Cyclamen*, which gave a blue colouration with iodine alone, and Schleiden[4] who discovered a similar phenomenon in the horny cells of the cotyledons of *Schotia*, *Hymenaea*, *Mucuna*, and *Tamarindus*, appropriated to the substance which produced the reaction the name Amyloid. Other instances of cells which turn blue, are afforded by the phloem cells of *Lycopodium*[5], and of the root of *Ruscus aculeatus*[6], the endosperm cells of *Paeonia*[7], *Ardisia* and *Primula*[8], and certain forms of mucilage, e.g. the mucilage cells of Linseed and Quince seed, and of the parenchyma cells of the peticle of *Aucuba Japonica*[9].

In some of the foregoing instances as in the case of certain young cells observed by Scharcht[10], a peculiar hydratic condition of the substance of the cell-wall is required, before the production of the blue coloured with iodine and sulphuric acid. To Scharcht also is due the observation which I have myself repeatedly confirmed with regard to endosperm cells, that very dry tissue will not turn blue until a certain amount of hydration has taken place. Finally Solla[11] noticed that in the young cells of the growing points of Zea, Phaseolus, and Vicia, the cell-walls do not turn blue, but simply remain yellow or brown, when treated with the usual cellulose reagent.

Of gums and mucilages one can say but little. There seems some

[1] Frémy, *Ann. de Sci. Nat.* Ser. 6, XIII. 1882.

[2] Berg, Zur Kenntniss des in Cetraria islandica vork. Lichenins und iodbläuenden Stoffes. Diss. Dorpat, 1872.

[3] Von Mohl, *Vermischte Schriften*, p. 335.

[4] Schleiden, *Grundz. der wiss. Botanik.* 3rd edition, I. p. 172, &c.

[5] De Bary, *Vergl. Anatomie*, p. 364.

[6] Gardiner. Resembles in reaction the phloem of Lycopodium.

[7] Vines (unpublished observation). Mentioned as such in my paper in the *Phil. Trans.* 1883. The phenomenon is also referred to in Trelease's English translation of Poulsen's *Microchemie*, p. 174, 1884.

[8] Gardiner, *Phil. Trans.* Part III. 1883.

[9] Gardiner. See present paper.

[10] Scharcht, *loc. cit.* p. 72.

[11] Solla, *Österr. Bot. Zeitschrift*, 1879, p. 351.

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probability that gums may be regarded as altered mucilages, i.e. as mucilages which have undergone some change in their properties, both in consequence of free exposure to the atmosphere, and of forming certain definite chemical combinations such as occur in gum arabic, which may be regarded as consisting of the arabates of calcium and potassium. Moreover, speaking generally, mucilages are connected with living, and gums with dead cells. One cannot however, I think, attempt to separate sharply the one class of bodies from the other. Between the typical gum and the typical mucilage, most distinct differences may exist, but there are, connecting the two extremes, many transition forms, which completely and insensibly bridge over the space and prevent us from placing with any degree of definiteness, a strict line of demarkation between the two. In any case, the bodies in question are to be regarded as mainly the result of a degeneration of the cell-wall. In gum-tragaganth the remains of the latter may be detected which moreover gives a blue with iodine and chlor. zinc. iod., but Cherry gum, on the other hand, displays no structure, and with the same reagent, simply stains yellow. With the mucilages which turn blue with iodine we shall deal later on.

In most cases we have certain microchemical tests, which afford a means of distinguishing the various modifications of cell-wall from one another, and since I may subsequently have occasion to refer to them it would be well that I should give them here.

Cellulose. Payen[1], as I have already mentioned, first demonstrated that walls containing cellulose gave a blue colour with iodine and sulphuric acid, and subsequently it was shown that the same reaction took place with iodine and zinc chloride (chlor. zinc. iod.)[2] soluble in sulphuric acid, ammoniacal oxide of copper, or Schweitzer's reagent[3] (copper, ammonio-sulphate), it is insoluble in Schultze's mixture (potassic chlorate and nitric acid), and indeed in oxidizing agents in general. Lastly, as I have pointed out[4], it is especially stained by dilute alcoholic solutions of methylene blue[5].

Lignin. In contradistinction to cellulose, this substance is soluble in oxidizing agents, such as Schultze's mixture, nitric acid, chlorine water, potassic permanganate or chromic acid. Such reagents convert it into resinous acid bodies which are soluble in alkalies. In sulphuric

[1] Payen, *loc. cit.*

[2] With Russow's mixture of iodine and chlor. zinc. iod. cellulose walls do not turn blue but simply remain brown. See *Sitzber. der Dorpat. Naturf. Gesell.* September, 1883.

[3] Schweitzer, *Vierteljahrsschrift. Naturf. Ges. Zürich*, Bd. II. 1857.

[4] Gardiner, *Phil. Trans.* Part III. 1883.

[5] With regard to the manipulation of this dye, it is important, that a dilute solution be made in 50 % alcohol; that the staining be quickly accomplished, and that the section be well washed in *water* before mounting in Glycerine. This treatment also applies to Hofmann's blue.

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acid or Schweitzer's reagent it is quite insoluble. It is deeply stained by most aniline dyes as Russow[1] showed for instance in the case of fuchsin. The two reagents discovered by Wiesner, viz. a solution of phloroglucin[2] or anilin chloride[3] in hydrochloric acid afford very valuable tests for lignified tissue, the former colouring it red or violet, and the latter staining it gold-yellow. With Von Höhnel's reagent[4]—xylophylin and hydrochloric acid—it also goes violet.

Mucilage. The reactions as regards mucilage are mostly of a negative character. With Hofmann's blue, (as I shall point out later on,) and with methylene blue[5] it readily assumes a blue colouration. With Hanstein's reagent[6] it stains red, and when well developed Roussin's test[7] (a white precipitate with ferrous sulphate, soluble in acetic acid) may be applied.

Cutin. The cutin of cuticle, like cellulose is insoluble in sulphuric acid, but soluble in caustic potash. With Hanstein's reagent it is coloured blue, and with chlor. zinc. iod. yellow.

Cork is dissolved by boiling potash, and during the process yellow oily drops (suberin of Von Höhnel[8]) escape. Subjected to the action of nitric acid, suberic acid is produced. With Schultze's mixture it is decomposed, yielding an oily, resinous mass, which may be dissolved by subsequent treatment with potash. According to Olivier[9] cork is readily stained by dilute solutions of fuchsin, which persistently colour it, even after prolonged action of absolute alcohol. With iodine and chlor. zinc. iod. it merely becomes yellow. When treated for some time with chromic acid it is dissolved.

An examination of the foregoing list will make it quite apparent that with regard to certain of the substances met with in the cell-wall, e.g. cellulose and cutin, there are well-defined chemical tests, which, on the whole, render it probable that the reactions to which they give rise, point to the presence of definite bodies, and are not merely the result of a histological differentiation. But as to many other reactions, which have also been regarded as evidences of the existence of other bodies, whose characters are not nearly so well defined, one must, for the present

[1] Russow, *Sitzber. d. Dorpater. Naturf. Gesell.* 1880, p. 419.
[2] Wiesner, *Sitzber. d. k. Akad. in Wien*, LXXVII. 1878.
[3] Wiesner, *Sitzber. d. k. Akad. in Wien*, LXX. 1874.
[4] Von Höhnel, *Sitzber. d. k. Akad. in Wien*, 1877. Wiesner believes that xylophylin is practically the same body as phloroglucin. According to Max Singer, *Sitzber. d. k. Akad. in Wien*, LXXXV. 1882, both Wiesner's and Von Höhnel's reactions are due to the presence of vanillin in lignified tissue. Thus they do not point to the presence of lignin, but rather of lignified tissue. The same is true of the indol advocated by Niggl. See Singer (*l. c.*). Wiesner's (*l. c.*) Phenol-Hydrochloric acid reaction is due to the presence of coniferin.
[5] Gardiner, *loc. cit.*
[6] Hanstein, *Bot. Zeit.* 1868. No. 43 et seq.
[7] See Bonnier. *Ann. des Sci. Nat.* 6th series, VIII. 1879, p. 87.
[8] Von Höhnel, *loc. cit.*
[9] Olivier, *Bull. Soc. bot. de France*, 1880, t. XXVII. pp. 234, 235.

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at any rate, regard them with suspicion, recognizing as we do the great difficulties connected with the isolation of pure substances, and consequently the little that is absolutely known of their chemical properties and constitution. This is perhaps especially true of the various bodies produced by the mucilaginous degeneration of cellulose.

It may be very justly observed that great light has been thrown on the constitution of the cell-wall, in consequence of two communications which have recently appeared in the "*Annales de Sciences Naturelles*," the one by Frémy[1], and the other by Frémy and Terreil[2]. By a careful and patient study of the chemistry of tissues, the well-known investigator who had so long ago isolated cutin[3], has now succeeded, not only in confirming his previous researches, but also in extending them, in such a manner, as to make a very decided and important advance in the field of Botanical Microchemistry.

The following is the classification adopted by the authors. Having it must be premised (though not definitely stated in their paper) separated the foreign substances occurring as cell-contents, &c. by some such method as successive treatment with petroleum spirit, alcohol, ether, and water, they arrange the various constituents of vegetable tissue under seven heads, e.g. (1) Cellulose substances, (2) Vasculose (Lignin), (3) Cutose, (4) Pectose, (5) Calcium Pectate, (6) Nitrogenous substances, (7) Mineral substances.

The cellulose substances include all those bodies which dissolve without colouration in bihydrated sulphuric acid (H_2SO_4, $2H_2O$) producing dextrin and sugar, and which resist the action of alkalies and oxidising agents. By means of Schweitzer's reagent (ammonio-sulphate of copper) three well-defined varieties may be separated.

1. *Cellulose proper.* Easily soluble in Schweitzer's reagent.
2. *Paracellulose.* Soluble only after the action of acids.
3. *Metacellulose.* Insoluble even after preliminary action of acids.

Vasculose (Lignin)[4]. Insoluble in bihydrated sulphuric acid or

[1] Frémy, *Ann. des Sci. Nat.* 6th ser. XIII. 1882.
[2] Frémy et Terreil, reference as before.
[3] Frémy, *Ann. des Sci. Nat.* 4 ser. XII. p. 331.
[4] Max Singer (*Sitzb. d. k. Akad. in Wien*, LXXXV. 1882) finds that after very lengthy treatment (e.g. 6 weeks) with water which was allowed to boil for 10 hours a day, the following substances may be extracted from lignified tissue, viz. (1) Vanillin, (2) Coniferin, (3) a certain quantity of gum, (4) the wood gum of Thomsen, (5) a substance coloured yellow by hydrochloric acid. From these results he is inclined to believe that the so-called lignin is in reality a mixture of many substances. It is not however apparent that by his methods he has actually extracted the lignin from the tissue. One would rather imagine the substances dissolved by boiling water were bodies which occur in lignified tissue. Thus on

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Schweitzer's reagent, but soluble in alkalies when subjected to increased pressure, and also dissolved by oxidizing agents as already mentioned, forming resinous acids which saponify with caustic potash.

Cutose (Cutin). Insoluble in bihydrated sulphuric acid or Schweitzer's reagent, but soluble in alkalies at ordinary pressure. Is converted by nitric acid into suberic acid.

By taking advantage of these properties, the various substances may be separated from one another. Thus cellulose may be separated from vasculose by acting upon the tissue either with bihydrated acid or with Schweitzer's reagent, while vasculose may be removed by the action of cold nitric acid diluted with an equal bulk of water, and subsequent treatment with alkali. Cutose in the same way is soluble in alkali at ordinary pressure.

Pectose. Insoluble in water, but converted into soluble pectin by the action of acids.

Calcic pectate. Decomposed by acids, e.g. dilute cold hydrochloric acid. The pectic acid which remains gives with potash a pectate soluble in water.

Nitrogenous substances. Soluble in potash.

Inorganic substances. Present in ash.

As to the occurrence of these substances the authors state that

Paracellulose is found in epidermal cells in the layers next the cuticle, and in certain cells of roots and other tissues.

Metacellulose occurs in the tissue of lichens and fungi. The fungin of Braconnot[1] consists largely of this form of cellulose.

Vasculose. In all lignified tissue.

Calcic pectate forms the chief constituent of the middle lamella and adjacent layers of many pith cells. In such tissue, a treatment with cold dilute hydrochloric acid causes separation from one another, of the constituent cells.

Nitrogenous substances embrace probably the remains of the protoplasm, and in the *inorganic salts* are included such substances as silica and the like.

the whole, one would suppose that the gums are derived from the cellulose framework, and that the coniferin, which as a glucoside exists probably as a cell-content of wood cells, escapes upon the death of the cell—either naturally or artificially induced—into the cell-wall; and thus causes a staining of that structure, just as in the case of cells containing tannin. The vanillin one would suppose is produced by the subsequent oxidation of the coniferin. It is therefore perhaps, to be regretted that in his translation of Poulsen's *Michrochemie* (Boston, 1884), Prof. Trelease has taken no account of the opinions of Dragendorff and Frémy, but has stated that "lignin probably consists of (1) vanillin, &c." If we are to regard lignin as the whole collection of the substances, other than cellulose, present in lignified tissues, then Frémy's new name of vasculose—which was otherwise unnecessary—is of value to distinguish the substance *par excellence*, which at the present time we usually admit to be present in woody tissue.

[1] Braconnot (*loc. cit.*).

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Among other valuable points we may notice that there is some definite evidence that the walls of fungi consist of a framework of cellulose and a large remaining portion of metacellulose. Again the suberin of Chevreul[1] is shown not to be a definite body, but a mixture of cutose and vasculose, and the suberic acid of Doepping[2] is derived from the cutose: the vasculose giving rise to the resinous acids. Thus cork consists of a cellulose framework and a remaining portion of cutose and vasculose, and we at once understand how that by the action of such a powerful oxidizing agent as Schultze's mixture, all but the cellulose framework is rendered soluble. It is of interest also to note that when heated with potash, cellulose gives rise to acetic and oxalic, and vasculose to ulmic acid.

But although one must be struck by the great value of the researches of Frémy and Terreil, one cannot I think admit with the former of these investigators, either that by their method a complete analysis of vegetable tissues can be made, or that their various subdivisions include all the substances met with in plant cells. One must regard the research rather in the light of an advance; very definite and very valuable, but still far from complete. For instance, it does not appear under what head that peculiar form of cellulose giving a blue with iodine, is to be classed, and especially with regard to the gums and mucilages, it is a question whether they can all be collected under the head of Pectose. As for lignin indeed, there seems to be great probability that, like cellulose, it is a definite substance, but concerning gums and mucilages there is every evidence that we have to deal with a class of bodies, the members of which although not very distinct, possess nevertheless many well-marked properties, of sufficient value to admit of their being clearly separated, and in fact, almost every well conducted analysis brings to light fresh forms of mucilage, which differ markedly in some well-defined reaction from the already existing varieties. It now remains for me to dwell shortly on some other points with regard to cellulose, lignin, cutin, and mucilages.

With the properties of what one may speak of as cellulose proper we are already acquainted. This body exists in an especially pure form, in many palm endosperms, e.g. *Bentinckia*, *Phoenix*, &c., and as I have already remarked, usually constitutes the substance of new and unaltered cell-walls. As to how paracellulose and metacellulose differ from the cellulose *par excellence* we do not as yet know, and whether the difference consists in oxidation, hydration, or any other change, must for the present be left an open question. Apparently, as far as one can judge

[1] Chevreul (*loc. cit.*). [2] Doepping (*loc. cit.*).

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from Frémy's paper, the paracellulose forms in great part the substance of the cuticularised layers of the epidermis, and since, as we saw, it is also met with in other tissues, one must conclude that this form of cellulose is of more general occurrence than has hitherto been imagined. Some interest also is attached to that particular form which turns blue with iodine alone. It has been long known that after the action of such acids as sulphuric, phosphoric, or iodic acids, cellulose walls will give a blue instead of a yellow, with iodine. In enquiring into the meaning of this phenomenon, we receive great assistance from the many investigations made under the directions of Prof. Dragendorff, which have special reference to the constitution of pure cellulose.

Although the formula of cellulose is usually regarded as $C_6H_{10}O_5$, Stackmann[1] found that an analysis of the cellulose of conifer-wood gave for the formula $5(C_6H_{10}O_5) + H_2O$. These results agreed with those of Koroll[2] on the constitution of the cellulose of sclerenchymatous and bast tissue. Similarly the cellulose of parenchyma cells gave $5(C_6H_{10}O_5) + 2H_2O$, while that from the wood of most Dicotyledons gave $5(C_6H_{10}O_5) + 3H_2O$. In all these analyses sulphuric acid was used in the purification. Schuppe[3] on the other hand, working his purification in the absence of sulphuric acid, found that the body thus separated was in all cases simply represented by $C_6H_{10}O_5$. As a result of these experiments it would then appear, that the use of sulphuric acid occasions a definite hydration of the cellulose, in the same way as we know it does in other chemical reactions[4].

Leaving for a moment the subject under immediate consideration, we find that Nägeli[5] and Sachsse[6], in their analysis of starch, found that the body should not be represented by $C_6H_{10}O_5$, but that the constitution expressed by the formula $6C_6H_{10}O_5 + H_2O$ was much more exact. Then comparing together, the facts with regard to the composition of starch, and of cellulose, it appears in the light of Schuppe's results, that the action of sulphuric acid and the like, brings about a definite hydration, in consequence of which a hydrated cellulose is produced, which, though it is most probably not identical with starch, yet approaches it in its constitution and, what is more, gives a blue colour with iodine. The cellulose met with in the asci of Lichens—(the so called Lichen-

[1] Stackmann, Studien über die Zusammensetzung d. Holzes. Diss. Dorpat, 1878.

[2] Koroll, Quant. chem. Unters. u. d. Zusammensetz. d. Kork-Bast, &c. Diss. Dorpat, 1880.

[3] Schuppe, Beiträge z. Chemie d. Holzgewebes. Diss. Dorpat, 1882.

[4] The production of ethyl alcohol by the action of sulphuric acid on ethylene is a case in point.

[5] Nägeli, *Annal. d. Chem. u. Pharm.* CLXXII. 218, 1874.

[6] Sachsse, *Zeitschr. f. anal. Chem.* XVII. 231, 1878.

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starch), the mucilage of Linseed, and the phloem cells of *Lycopodium*, and of the root of *Ruscus aculeatus* must, probably, be regarded as consisting of an hydrated cellulose naturally produced, whereas with regard to the cells which turn blue only after the action of acid, such hydration is induced artificially.

There are some definite grounds also for believing that the so-called fungus cellulose (metacellulose of Frémy) consists of a form of hydrated cellulose, since Masing[1] showed that in the conversion of cellulose into glucose, the cellulose of fungi underwent the change much more rapidly than that of flax fibre. We know of course that this conversion is simply a process of hydration.

Just as cellulose walls consist mainly of a definite body—cellulose—so it seems probable from the researches of Frémy, and in the opinion of such a well qualified chemist as Dragendorff[2], that in lignified tissue a definite substance—lignin, occurs. At present however we have but few facts to go upon. The difficulties attending its isolation are so great, that it appears somewhat questionable whether the real body has as yet been obtained pure, and unacted upon by reagents, and knowing as we do the difficulties attending the separation from it, of foreign substances, we must as yet regard with some doubt, researches such as those of Erdmann[3] and Bente[4] on pine wood, which would tend; either to show that in lignified cell-walls the cellulose and lignin are chemically combined as glycolignose; or that lignin, as such, actually yields pyrocatechin as a product of its decomposition; thus pointing to its relations with the benzol series. In fact although certain of its reactions may, and probably do, point to the existence of a definite body yet that body has most probably not yet been isolated.

Of cutin one has but little to say. As has been already remarked the suberin of Von Höhnel has been shown by Frémy to consist of a mixture of cutin and lignin, and that it is from the cutin that the suberic acid is derived. In corky as in lignified cells a framework of cellulose is always present which although it may or may not be directly obvious, can always be shown to exist after the action of Schultze's mixture in which both the lignin and cutin are soluble[5].

Dealing now with mucilages one must confess at the outset that the chemistry of these substances is still in a most unsatisfactory state. Nor is this to be wondered at. Leaving out of

[1] Masing, *Pharm. Zeitschr. f. Russland*, IX. 385, 1870.
[2] Dragendorff, *loc. cit.* p. 253.
[3] Erdmann, *Annal. d. Chem. u. Pharm.* CXXXVIII. 1, 1866.
[4] Bente, *Ber. d. d. chem. Ges.* XIII. 476, 1875.
[5] Lemaire, *Ann. Sci. Nat.* XV. (1883), p. 297, finds that in certain cases, e.g. the Cycadeae, the epidermis itself has undergone lignification.

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the question the difficulties connected with the mere analysis of these organic compounds; any one at all acquainted with chemical methods, will at once understand how hard it is to deal with even the ordinary manipulative details of such colloidal bodies as mucilages, where even the washing, and filtration, offer almost insuperable obstacles to the obtaining of a pure product.

We will first consider the gums and the pectous substances. Chemists have taught us that the gums, as represented for instance by gum-arabic, usually consist of the potassium and calcium salts of a weak acid: arabic acid or arabin. By the action of dilute acids upon the body; metarabic acid, and finally a glucose (arabinose) is formed[1]. Another variety was described by Reichardt[2] which is distinguished in that it does not give arabinose as a result of the acid of an acid.

From the mucilage occurring in the pulp of fruit, a substance—pectose—has been extracted, which by the action of acids gives soluble pectin, and further by the action of the peculiar pectose ferment described by Frémy, or simply by a regulated treatment with dilute alkalies and acids, gives rise to pectic, parapectic and finally metapectic acid.

The remarkable similarity between the properties of Arabic acid on the one hand, and Pectic on the other, appear to point to the probability, that the two bodies are identical, and in the opinion of Reichardt and Dragendorff and (since he puts all the gums and mucilages under the head of Pectose) of Frémy, the pectin substances may in fact be regarded simply as varieties of the mucilages and gums. Of the mucilages in particular, the varieties appear to be very numerous, for in addition to Berg's discovery of Lichenin which I have already mentioned, Kirchner[3] believes he has obtained a pure mucilage from Quince seeds, and Thomsen[4] from ligneous tissue has extracted a wood-gum. Gelose[5] which is related to Lichenin, has been found to occur in many algae, and more recently from *Fucus Amylaceus* has been separated a substance which though allied to Gelose and Lichenin differs markedly from them both[6].

From *a priori* considerations, it does not seem improbable that certain kinds of mucilage may be regarded as formed from the cellulose in consequence of excessive hydration; but since, as

[1] According to Scheibler, *Ber. d. d. chem. Ges.* VI. 620, two sugars (one crystallisable and the other uncrystallisable) and an acid whose barium salt is insoluble in alcohol, are produced.

[2] Reichardt, *Ber. d. d. chem. Ges.* VIII. 807, 1875.

[3] Kirchner, *Ueber Pflanzenschleim.* Diss. Inaug. Göttingen, 1874.

[4] Thomsen, *Journ. Pract. Chem.* (2) XIX. 146, 1879.

[5] Morin and Porumbaru, *Comptes rendus*, XC. 924, 1081, 1810.

[6] *Sitzber. Naturfors. Ges. Dorpat*, VI. p. 39, 1881.

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Wigand[1] showed, mucilage may also be produced by the disorganisation of starch-grains and, as Trecul[2] and Prillieux[3] found, even from the protoplasm itself, we must not carry this assumption too far. It was an interesting observation made by Prillieux that in the gum-producing cells of the cherry-tree, the substance in question first appeared between the middle lamella and the last layer of cell-wall, thus resembling the similar phenomenon which occurs in connection with the appearance of oil, &c. beneath the cuticle in glandular hairs and the like as described by Personne[4] in the hop, by Cohn[5] in *Siphocampylus*, by Batalin[6] in *Sarracenia*, and more recently in the extended observations of Martinet[7], Bonnier[8] and Behrens[9].

Gums and mucilages occur in all parts of the plant. In many of the Leguminoseae, the gum, e.g. gum-tragaganth, is produced by the disorganisation of the cell-walls of the pith and medullary-rays[10], and in the Rosaceae, e.g. the cherry, it is formed both in the cortex, the medullary rays, and even from certain of the parenchyma-cells of the vascular tissue[11]. The mucilage of seeds is usually derived from the middle layers of the thickened epidermal cells, but it also occurs in cells containing raphides; in definite passages as in the Marattias[12], from rows of cells as in the Marchantias[13]; or according to the interesting observation of de Bary[14] is secreted by certain hairs at the base of the petiole of Osmunda. The mucilaginous degeneration undergone by the outer layers of the walls of so many algae has long been known, but has recently been prominently brought into notice in consequence of the observations of Schmitz[15] upon the surface growth of cells. It is also a matter of some interest that in the parenchyma cells of many ordinary vegetable tissues the same phenomenon

[1] Wigand, Ueber die Desorganisation der Pflanzenzelle, &c. Pring's *Jahrb.* III. 115, 1863.

[2] Trecul, *Procès Verbal des Séances de la Société philomatique pendant l'année*, 1862.

[3] Prillieux, *Ann. des Sci. Nat.* ser. 6, I. 1875.

[4] Personne, *Ann. des Sci. Nat.* ser. 4, I. 1842.

[5] Cohn, *De cuticula*, Wratislaviæ, 1850.

[6] Batalin, *Ueber die Function der Epidermis von Sarracenia*, &c. St Petersburg, 1880.

[7] Martinet, *Ann. des Sci. Nat.* ser. 5, XIV. 1871.

[8] Bonnier, *Ann. des Sci. Nat.* ser. 6, VIII. 1879.

[9] Behrens, *Flora*, 1878, 1879.

[10] Von Mohl, *Bot. Zeit.* 1857.

[11] Prillieux, *loc. cit.*

[12] Harting et de Vriese, *Monogr. des Maratt.*

[13] Prescher, Die Schleimorgane der Marchantieen. *Sitzber. d. k. Akad. in Wien*, LXXXVI. 1882.

[14] De Bary, *Vergl. Anatomie*, p. 105. First mentioned by Milde, *Monogr. Generis Osmundae.*

[15] Schmitz, *Sitzber. niederrhein. Ges. in Bonn*, Dec. 6th, 1880. See also *Sitzber. Versammlung deutscher Naturf. u. Aerzte in Eisenach*, Sept. 19th, 1882.

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occurs. The conversion of a portion of the cellulose substance into mucilage may take place in such a manner, that the degeneration is almost entirely limited to the middle lamella, or it may extend to several layers of the wall, or finally, mucilage may be detected throughout the entire substance of the cell-membrane.

The phenomenon of mucilaginous degeneration was strikingly brought to my notice during my research upon the continuity of the protoplasm through the walls of vegetable cells[1]. As I have already stated in my papers upon that subject, I found that Hofmann's aniline blue was an especially good stain for the protoplasm, and I have in consequence used it for staining the fine protoplasmic threads which traverse the cell-wall. In the course of the investigation I repeatedly observed, that in many cases a very marked colouration took place, not only of the protoplasmic threads, but also of the most external layers of the cell-walls, and in certain instances, e.g. *Chara foetida* and the endosperm cells of *Tamus communis*, the whole of the walls were distinctly blue. The colouration was usually not well defined, but was darkest next the middle lamella and gradually faded off towards the cell-lumen, as for example, in most collenchymatous hypodermal cells, and the cortical cells of the petiole of *Ilex aquifolium*, *Aesculus hippocastanum*, &c. In some exceptional instances however, e.g. *Aucuba Japonica*, a very definite staining did occur, of what one might speak of, as a well-defined intercellular substance, which could hardly be distinguished from the almost similarly stained protoplasm. I had intended to investigate the subject further, and on that account did not treat of it in my paper in Sachs' *Arbeiten*[2], since my observations were still incomplete. Quite recently however the matter was again brought very forcibly to my notice, on account of certain passages which appear in a preliminary communication I received from Prof. Russow[3] on "the connection of the protoplasmic bodies of neighbouring cells," in which he announces the discovery of an intercellular protoplasm, between the cortical parenchyma cells of *Acer* and *Fraxinus* and in the same cells of the rhizome of *Iris*, &c.

In consequence of this I again renewed my investigations, and as a result, I find, that both Hofmann's blue and water blue (one or the other of which Russow most probably used[4]) stains not only

[1] Gardiner, *Arbeiten des botanischen Instituts in Würzburg*, Bd. III. Heft. I.

[2] Gardiner (*loc. cit.*).

[3] Russow, *Sitzber. der Dorpat. Naturfors. Gesell.* September, 1883. In this paper Russow confirms in a most striking manner, the results I have already obtained as to the communication of the protoplasmic contents of adjacent cells by means of the sieve-like perforation of the pit-closing-membranes. I have only in rare instances, e.g. parenchyma cells of Aucuba Japonica, seen the nodular swellings of the connecting protoplasmic threads, of which he speaks.

[4] See Gardiner, *Phil. Trans.* p. 829.

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the protoplasm; but also mucilage, and that in the case of my own and Russow's observations, the staining was simply due to the mucilage in question. The whole phenomenon of mucilaginous degeneration leading to the production of the intercellular mucilage, may perhaps be most easily followed in longitudinal sections of the petiole of *Aucuba Japonica*. In the cells with large intercellular spaces, numerous drops of mucilage, in the form of small papillae may be observed on the outer or free surface of the cell-walls, which are not only distinguished by their microscopical appearance before they aggregate to form a layer, but will like the similar substance in Quince seeds, go blue with iodine, owing to the presence in their structure of a definite hydrated cellulose.

It may be imagined that in consequence of the facts in connection with the staining of Hofmann's blue that the observations based upon the reactions of this substance lose much of their value, and that from *a priori* considerations the so-called protoplasmic threads may as well be mucilage as protoplasm. Such however is not the case. For the results as to the protoplasmic character of the threads do not depend upon the reactions of Hofmann's blue alone, but iodine and chlor. zinc. iod., and the sulphuric and molybolic acid mixture were also employed. Moreover it is quite possible to experiment upon walls which both consist entirely of pure cellulose, free from mucilaginous degeneration, and in which the middle lamella is not conspicuously developed. Thus in the endosperm cells of *Bentinckia Condapanna* and *Latania Loddigesii* which fulfil both of these conditions, the only staining which occurs, is sharply limited to the protoplasm and to the threads, and the results are entirely confirmed by chlor. zinc iod., the reactions of which, towards protoplasm and mucilage, admit of being clearly distinguished. Lastly, all doubt upon the subject is put an end to by the use of methylene blue. This substance, as I have pointed out, stains markedly the cell-walls and all the substances produced by the degeneration of the same, such as mucilage, &c. On the other hand methylene blue does not (unless the staining be forced) colour the protoplasm. So that whereas Hofmann's blue stains protoplasm and mucilage, but not cell-wall; methylene blue stains cell-wall and mucilage but not protoplasm, and in this way the two may be clearly separated.

If this be so, it will then be of interest to observe the staining effects of these two reagents upon the callus of sieve-tubes. As regards this substance Janczewski[1] maintains that it is pre-eminently of a mucilaginous nature, while Russow[2] and myself[3] on

[1] Janczewski, *Mém. de la Soc. des Sciences Naturelles et Mathematiques de Cherbourg*, t. XXIII. p. 209.
[2] Russow, *Sitzber. der Dorpater Naturfors. Gesell.* Feb. 17th, 1882.
[3] Gardiner (*loc. cit.*). See also *Phil. Trans.* Part III. 1883.

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the other hand, believe that it is rather allied to protoplasm. As we know from Russow's results, the callus is stained by anilin blue (either Hofmann's blue or water blue). This fact may as I have showed point either to its protoplasmic or its mucilaginous character. But with methylene blue no staining whatever occurs, which not only absolutely negatives the probabilities of its mucilaginous character, but also gives a most valuable confirmation to the results which I have already set forth. The very deep staining of the callus with dilute Kleinenberg's haematoxylin also points to its protoplasmic nature[1].

As regards its delicacy, and its power of clearly defining the callus, Russow's[2] mixture of iodine in potassium iodide and chlor. zinc. iod. undoubtedly occupies the first place. Then comes Hofmann's blue and haematoxylin, and lastly the rosolic acid test of Szyszylowicz[3], the staining characters of which are not nearly so distinct, and its properties, so far as I am aware, can only be observed when the section is mounted in a solution of the reagent; disappearing entirely when the section is washed in order to mount in water, or what not, so as to obtain the greatest amount of differentiated staining.

Returning now to the main subject under consideration, we see that the mucilaginous degeneration of the cell-wall is of very frequent occurrence, and that the relative proportion existing between the unaltered cellulose and the mucilage may be very variable. In almost all cases however, as Kirchner[4] showed for example in the mucilage of Quince seeds, a definite residuum of cellulose occurs, which may, as in Tragaganth gum, make itself evident by its reactions with chlor. zinc. iod., or as in cherry gum, cannot (michrochemically at least) be detected.

We are now in a position to deal with the constitution of the middle lamella. To the earlier botanists this structure was known as the intercellular substance, which appeared to be excreted by the cells, and thus formed the common matrix in which the cells were imbedded; but at the present time we regard it as consisting of the first formed cell-wall together with one or more

[1] Besides the callus, haematoxylin also stains the refringent granules. The dense protoplasm (slime) of sieve tubes is especially coloured by eosin which however does not stain the callus. From what occurs in Vitis when examined in the winter and summer condition—the sections being stained with eosin—there seems to be some definite grounds for believing that the callus is formed from the dense protoplasm. Whether the latter is formed by the coalescence of the refringent granules, must for the present be left an open question.

[2] Russow, *loc. cit.*, Feb. 17th, 1882.

[3] Szyszylowicz, *Osobne. odbicie z Rospran Akad. Umiej. w Krakowie*, x. (1882). See *Bot. Central.* XII. (1882), p. 138.

[4] Kirchner, *loc. cit.*

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layers, as the case may be, which during the subsequent thickening, have become physically and chemically altered[1].

In the light of Schmitz[2] and Strasburger's[3] researches, it will be readily understood that the formation of such a differentiated portion in the thickening cell-wall, is naturally necessitated, on account of the amount of tension and pressure produced during the process of surface growth, which although it may be small in the last or most internal layer deposited, in the region of the middle lamella reaches its maximum, and becomes very appreciable. As regards the chemical phenomena, we know that it is admitted on all hands that the various changes accompanying lignification, cuticularisation, &c. are most marked in the more external layers of the cell-wall, and in consequence we find that the substance of the middle lamella has usually undergone the maximum amount of alteration from its primitive cellulose character, so that both chemically and physically it may be truly regarded as the most modified portion.

From the standard text-books generally in use, one is apparently led to infer that the middle lamella consists of a certain definite substance—the substance of the middle lamella—which is distinguished by the fact that it is insoluble in sulphuric acid, but soluble in Schultze's mixture. From these properties such a substance might consist either of lignin or cutin or finally of cork which we regarded as a mixture of both, for all these substances would be soluble in Schultze's mixture. There is therefore great want of definiteness in the statements with regard to this substance, which has been thus separated in botanical terminology, and I shall therefore endeavour by means of the evidence afforded by direct observation, and by recent research, to state the matter somewhat more clearly.

In certain cells, e.g. bast prosenchyma cells of the pulvini of *Mimosa* and many palm endosperms, the cell-walls consist of pure cellulose, and the middle lamella is but little developed. In such cases one finds that the middle lamella although more resistant than the rest of the cell-wall in consequence apparently of its greater density is distinctly soluble in sulphuric acid, and as far as one can observe consists simply of cellulose. In other instances, e.g. lignified prosenchyma cells of the cortex of *Lycopodium*, the well-defined middle lamella equally with the layers of the cell-wall assumes a gold-yellow colour when treated with aniline chloride and hydrochloric acid, and with any other test

[1] This latter fact was insisted upon by Solla, "Beiträge zur näheren Kenntniss der chemischen und physikalischen Beschaffenheit der intercellular Substanz. (*Œsterreichische bot. Zeitschrift*, November, 1879.)

[2] Schmitz, *loc. cit.*

[3] Strasburger, *Zellhäute*, pp. 1—146 and 175—200.

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for lignified tissue they both stain in a similar manner. There can in fact be little doubt that here the middle lamella has undergone lignification. In the same way in suberised or mucilaginous membranes, the middle lamella undergoes the same changes as the cell-walls. With regard to the mucilaginous change, it may be noticed that the middle lamella may be at once converted into mucilage as in the parenchyma cells of the petiole of *Aucuba*[1]; or as Prillieux[2] remarked in the formation of gum in the cherry, the lamella may be at first dense and resistant, in which case the mucilage appears between it and the last layer of cell-wall. Consequently it only subsequently becomes mucilaginous.

The great point with regard to middle lamellas other than cellulose is that in their substance the maximum amount of change appears to have taken place—that is to say—almost the whole of the cellulose has been converted into lignin, cutin, or mucilage, as the case may be, and thus but little of the cellulose framework remains. This will of course explain the fact that after treatment with Schultze's mixture or other oxidizing agents, the various cells readily separate from one another, for now practically the whole of the middle lamella has suffered solution, and, of the cells, it is only the cellulose framework which remains.

It would thus appear that in unaltered cellulose walls the middle lamella consists of dense cellulose; while in lignified cuticularised corky or mucilaginous cells, the changes which occur in the middle lamella are of the same character as those of the rest of the membranes, and have moreover reached their maximum.

Additional Note.

I should like to add here a few remarks with regard to the continuity of the protoplasm through the walls of vegetable cells. Since my last publication upon that subject, a paper has appeared in the February number of the *Journal of Botany*[3] in which the writer states that in the Florideae there is a communication between the contents of adjacent cells, which so far as one can judge from the text, is typically brought about by means of open pits, although in certain instances, e.g. the older cells of Polysiphonia fastigiata, a delicate diaphragm may be present.

[1] In such instances the cells separate by mere treatment with hot water. The well known case of the separation from one another of the cells of Potato tubers when boiled is another example. When on the other hand the mucilage or gum is combined with bases, e.g. in case of Calcium Pectate, an acid is required to bring about the isolation of the cells.

[2] Prillieux, *loc. cit.*

[3] Hick, *Protoplasmic Continuity in the Florideae.* Since I find (March 17th) that in the remaining part of his paper, which appears in the March number of the same Journal, he has in no way altered his opinion, I have nothing to add to my former remarks.

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This being the case, it will be noticed that the author has simply repeated anew, the observations made in 1878 by Perceval Wright[1] and Thuret[2], and is apparently unacquainted with the researches of Schmitz[3], who has showed that except perhaps in the Corallines a pit-closing-membrane is present in each cell, which is usually perforated in a sieve-like manner. I have myself investigated several of the Florideae, and I find, that in all the cases which have come under my observation a distinct pit-closing-membrane can always be made out after proper treatment.

Of the existence of a continuity of the protoplasm between neighbouring cells there seems but little doubt, and indeed, so tenaciously do the protoplasmic processes of the pits adhere to the closing membranes that it is a matter of difficulty to demonstrate that such a membrane exists; the protoplasm refusing to separate even after the action of very strong plasmolysing agents. Thus in the Polysiphonias which may be taken as typical representatives, e.g. *P. nigrescens*, a treatment with strong sulphuric acid (as observed by Thuret[4]) or with strong salt solution, does not separate the protoplasmic processes of the pits from one another, but on the contrary every appearance of what one might conveniently speak of as a direct, unbroken continuity exists. When on the other hand, the fresh tissue is treated for some time with solutions of calcium chloride or chlor. zinc. iod., previous to preservation in alcohol a different appearance is produced. In the first case the protoplasmic processes of the pits contract to the main protoplasmic mass of the cell, and in so doing leave a distinct pit-closing-membrane which is swollen on both sides, giving to the whole structure a lenticular form and recalling the torus which occurs on the pit-closing membranes of certain bordered pits. After treatment with chlor. zinc. iod. this lenticular doubly convex body may also be left as a pit-closing-membrane, but in certain cases one can detect that the membrane in question is further resolved into a thin pit-closing-membrane, having on either side of it a small mass which now possesses a plano-convex form.

To explain these phenomena I have adopted the following view, which however I do not consider as final, since I have not investigated the subject as thoroughly as I could wish. The pits of the Florideae and consequently the protoplasmic processes which enter them rapidly narrow, from the pit-closing-membrane towards the cell-lumen, so as to present a trumpet-like shape. Of the processes themselves; that portion which abuts immediately on the closing membrane, both adheres with great tenacity to that struc-

[1] See Perceval Wright's two papers, *Trans. Roy. Irish. Acad.* xxvi. 1879.
[2] Thuret, *Études Phycologiques*, p. 100, 1878.
[3] Schmitz, *Sitzber. Akad. Wiss. Berlin*, pp. 215—58, Feb. 22nd, 1883.
[4] Thuret, *loc. cit.*

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ture, and also, on account of the rapid constriction of the pits, tends to be held firmly in position. When by plasmolysis, &c., one attempts to contract the pit processes to the main protoplasmic cell-mass, the particular portion next the pit-closing-membrane usually remains behind, and on account of its subsequent rounding off, presents the appearance of a lenticular body as induced for instance by the action of calcium chloride. But in other cases when, owing to the particular combination of forces, the whole of the pit-protoplasm endeavours to separate, then the delicate closing membrane becomes clearly apparent.

The portion of the pit-protoplasm next the pit-closing-membrane is usually well differentiated, and on that account, and on account of its peculiar form, has been distinguished both by Schmitz[1] and also by Perceval Wright[2], the latter of whom appropriates to it the name "stopper", and states that in every cell, at least two of these stoppers are present.

Thus so far as I have been able to observe, a pit-closing-membrane is always present, both in the cortical, and the central cells, which may be extremely difficult to bring into view, as in the Polysiphonias, but in certain other cases may be seen with comparative ease. There is little doubt that the closing membrane consists of cellulose, and that it is perforated in a sieve-like manner as Schmitz has already described for many of the Florideae. I also believe from my somewhat scant experiments with preserved material, that in the Corallines the same structure occurs as in the rest of the Florideae.

Passing to other matters I find that Wille[3] has observed a continuity of the protoplasm in *Stigonema compactum*, and I also, before I was acquainted with Wille's researches, found the same to occur in the case of *Nostoc*. In *Volvox globator*, so far as I have been able to investigate it, I believe that a similar continuity exists.

[1] Schmitz, *loc. cit.*

[2] Perceval Wright, *loc. cit.* The name was first employed by Archer. See his paper "On the minute structure and mode of growth of Ballia callitricha," *Trans. Linn. Soc.* ser. 2, I. p. 211.

[3] Wille, *Ber. Deutsch. Bot. Gesell.* I. 1883. Wille also finds a nucleus in the cells of Tolypothrix lanata, thus confirming Schmitz's discovery of the presence of a nucleus in the Phycochromaceae. If Guignard's statement with regard to the nucleus of the sieve-tubes of Vitis be confirmed (*Bull. Soc. Bot. France*, XXVIII. pp. 332—333, 1881) it can hardly be doubted that a nucleus is present in every living cell whatsoever.

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(2) *On the Head Kidney of Bdellostoma.* By W. WELDON, B.A.

The structure known as the "head kidney" was stated to consist of a bunch of branched tubules, opening on the one hand into the pericardium, and on the other into a central duct, which had lost its primitive connection with the segmental duct. At its posterior extremity blood was stated to enter the duct by means of vessels passing from a glomerulus.

The whole organ was compared to the suprarenal bodies of higher vertebrates, and it was suggested that the evidence at present obtained as to the embryonic development of the suprarenals rather confirmed than disproved the view that they were derived from parts of the primitive kidney.

Functionally, an attempt was made to compare the suprarenals of vertebrates with the glands always found, in one form or another, in connection with the circulatory system of invertebrata.

(3) *On the early stages in the development of Balanoglossus Aurantiacus.* By W. BATESON, B.A.

The author stated that through the great kindness of Dr W. K. Brooks and the Council of Johns Hopkins University he had been permitted to work during the past summer at the Chesapeake Zoological Laboratory. He had thus been able to observe some larval stages of Balanoglossus Aurantiacus (? sp.). The development of this form from the egg was stated to be a direct one, no form at all comparable to Tornaria being passed through. The gastrula was anal, and the mesoblast arose as five archenteric diverticula, while the central nervous system was described 'as being formed by the delamination of an epiblastic plate in the median dorsal line.' A brief comparison was made between the structure of this larva and that of Tornaria and the Echinoderm larvæ on the one hand and with the early development of Amphioxus on the other.

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February 25, 1884.

Mr GLAISHER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

A. R. Forsyth, B.A., Trinity College.
W. J. Ibbetson, B.A., Clare College.

The following communications were made to the Society:

(1) *On the sum of the divisors of a number.* *By* J. W. L. GLAISHER, M.A., F.R.S.

§ 1. Denoting by $\sigma(n)$ the sum of the divisors of n, it was proved by Euler that

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\ldots=0,$$

where 1, 2, 5, 7, ... are the pentagonal numbers given by the formula $\frac{1}{2}r(3r\pm1)$. The series is to be continued until the arguments become negative and the term $\sigma(n-n)$ or $\sigma(0)$, when it occurs, is to be replaced by n.

The term $\sigma(n-n)$ occurs only when n is itself a pentagonal number, and if we make no convention with regard to the meaning to be assigned to $\sigma(0)$, but suppose it to have its proper value zero, the theorem becomes

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\ldots$$
$$=0 \quad \text{or} \quad (-1)^{r-1}n,$$

according as n is not, or is, a pentagonal number $\frac{1}{2}r(3r\pm1)$.

This is the form in which the equation arises as the result of the process by which Euler obtained it. In the second case when $n=\frac{1}{2}r(3r\pm1)$ the expression on the left-hand side of the equation contains the term $(-1)^r\sigma(0)$, and by conventionally defining $\sigma(n-n)$, when it occurs, to denote n, we obtain the theorem in the first form, which is that in which Euler preferred to enunciate it. This theorem was the first one of its kind discovered, and it appeared to Euler to be of the very highest interest, as it afforded a method of calculating the sum of the divisors of a number (and thus also a means of deciding whether it was prime or not) by the sole aid of operations which have no relation whatever to the divisors themselves*.

§ 2. Euler deduced his formula from the equation

$$(1-x)(1-x^2)(1-x^3)\ldots=1-x-x^2+x^5+x^7-x^{12}-x^{15}+\&c.$$

* "Observatio de summis divisorum," *Opera Minora Collecta*, Vol. I. pp. 146—154.

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and, by applying his process to Jacobi's equation,

$$\{(1-x)(1-x^2)(1-x^3)\ldots\}^3 = 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c.,$$

in which the exponents are the triangular numbers given by the formula $\frac{1}{2}r(r+1)$, we find that

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - \ldots$$
$$= 0 \quad \text{or} \quad (-1)^{r-1}\tfrac{1}{8}r(r+1)(2r+1),$$

according as n is not a triangular number, or is a triangular number $\frac{1}{2}r(r+1)$.

Thus, for example, if $n=9$, which is not a triangular number, the formula gives

$$\sigma(9) - 3\sigma(8) + 5\sigma(6) - 7\sigma(3) = 0,$$

that is, $$13 - 3\times 15 + 5\times 12 - 7\times 4 = 0;$$

and if $$n = 10 = \tfrac{1}{2}\times 4\times 5 \text{ so that } r = 4,$$

it gives

$$\sigma(10) - 3\sigma(9) + 5\sigma(7) - 7\sigma(4) = (-1)^3\tfrac{1}{8}\times 4\times 5\times 9,$$

that is, $$18 - 3\times 13 + 5\times 8 - 7\times 7 = -30.$$

The quantity $\frac{1}{6}r(r+1)(2r+1)$ is the well-known expression for the sum of the first r square numbers; and, if $n = \frac{1}{2}r(r+1)$, the number of terms on the left-hand side of the equation is r. In general therefore when n is a triangular number, the series is numerically equal to

$$1^2 + 2^2 + 3^2 \ldots + r^2,$$

where r denotes the number of terms it contains, and the sign is the same as the sign of the last term. Thus, for example,

$$\sigma(10) - 3\sigma(9) + 5\sigma(7) - 7\sigma(4) = -(1^2 + 2^2 + 3^2 + 4^2).$$

This is however a merely curious form in which the result admits of being exhibited, but by adopting a convention of a similar kind to Euler's with respect to the meaning to be assigned to $\sigma(0)$ we may enunciate the theorem in a form which is very convenient in use and in which the right-hand member is always zero. For when the right-hand member of the equation is $(-1)^{r-1}\frac{1}{8}r(r+1)(2r+1)$, the expression on the left-hand side contains the term $(-1)^r(2r+1)\sigma(n-n)$. If then instead of putting $\sigma(n-n) = \sigma(0) = 0$ we replace it by $\frac{1}{8}r(r+1)$, the right-hand member of the equation becomes zero. Now in this case $n = \frac{1}{2}r(r+1)$, so that in fact we replace $\sigma(0)$ by $\frac{1}{3}n$. Thus we have, for all values of n,

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + \ldots = 0,$$

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where $\sigma(0)$, when it arises, is defined to denote $\frac{1}{8}n$. For example,

$$\sigma(10) - 3\sigma(9) + 5\sigma(7) - 7\sigma(4) + 9 \times \tfrac{1}{8}9 = 0.$$

§ 3. The following formula, which is of the same kind as Euler's but in which n is restricted to be of a particular form, was obtained by means of Elliptic Functions*.

If n be of the form $8m+7$, then

$$\sigma(n) - 2\sigma(n-4) + 2\sigma(n-16) - 2\sigma(n-36) + \ldots = 0.$$

For example, let $n = 55$, the formula gives

$$\sigma(55) - 2\sigma(51) + 2\sigma(39) - 2\sigma(19) = 0,$$

that is,
$$72 - 2 \times 72 + 2 \times 56 - 2 \times 20 = 0.$$

In this formula $\sigma(0)$ cannot occur.

§ 4. The formulæ given in the preceding sections are interesting, if only for the reason mentioned by Euler; but for the actual calculation of a table it is preferable to employ the equation

$$\sigma(n) = \sigma(n_1)\,\sigma(n_2)\,\sigma(n_3)\ldots,$$

where n_1, n_2, n_3...are prime to one another and $n = n_1 n_2 n_3 \ldots$. They would be useful in verifying a table of $\sigma(n)$, but, as the intervals between the terms are unequal, the verification afforded is not systematic, and it is not obvious how to apply them in order to verify by their means all the numbers in a table.

In seeking for formulæ which would afford a more complete verification of a table of $\sigma(n)$ I obtained also the following four formulæ:

(i)

If n be even, then

$$\sigma(1)\,\sigma(n-1) + \sigma(3)\,\sigma(n-3) + \sigma(5)\,\sigma(n-5) \ldots + \sigma(n-1)\,\sigma(1)$$
$$= \tfrac{1}{8} \times \text{sum of the cubes of those divisors of } n \text{ which have uneven conjugates.}$$

For example, putting $n = 6$,

$$\sigma(1)\,\sigma(5) + \sigma(3)\,\sigma(3) + \sigma(5)\,\sigma(1) = \tfrac{1}{8}(6^3 + 2^3),$$

that is,
$$6 + 16 + 6 = \tfrac{1}{8} \times 224.$$

* *Quarterly Journal of Mathematics*, Vol. xx. p. 121.

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(ii)

If $m = 2n + 1$, n being unrestricted, then

$$n\sigma(m) + (n-5)\,\sigma(m-2) + (n-15)\,\sigma(m-6) + (n-30)\,\sigma(m-12) + \ldots = 0;$$

the differences between the arguments in the successive terms being the even numbers 2, 4, 6, 8 ... and the differences between the successive multipliers being the multiples of five, viz. 5, 10, 15 ... . The series is to be continued until the arguments become negative, and there is no convention with regard to $\sigma(0)$, which, when it occurs, is to have its proper value zero.

For example, let $n = 7$ so that $m = 15$; the formula gives

$$7\sigma(15) + 2\sigma(13) - 8\sigma(9) - 23\sigma(3) = 0,$$

that is,
$$7 \times 24 + 2 \times 14 - 8 \times 13 - 23 \times 4 = 0.$$

(iii)

If $m = 2n + 1$ and $p = 4n + 1$, n being unrestricted, then

$$\psi(p) + \psi(p-8) + \psi(p-24) + \psi(p-48) + \ldots$$
$$= \psi(m) + 2\psi(m-2) + 2\psi(m-8) + 2\psi(m-18) + \ldots,$$

the numbers 8, 24, 48 ... being of the form $4r(r+1)$, and 2, 8, 18 ... being the doubles of the squares.

As an example, let $n = 2$; then $m = 5$, $p = 9$, and the formula gives

$$\psi(9) + \psi(1) = \psi(5) + 2\psi(3),$$

that is,
$$13 + 1 = 6 + 8.$$

(iv)

If $m = 2n + 1$ and $r = 4n + 3$, n being unrestricted, then

$$4\{\psi(m) + \psi(m-4) + \psi(m-12) + \psi(m-24) + \ldots\}$$
$$= \psi(r) + 2\psi(r-4) + 2\psi(r-16) + 2\psi(r-36) + \ldots,$$

the numbers 4, 12, 24 ... being of the form $2r(r+1)$ and 4, 16, 36 ... being the even squares.

Taking, as in (iii), $n = 2$, we have $m = 5$, $r = 11$ and the formula gives

$$4\{\psi(5) + \psi(1)\} = \psi(11) + 2\psi(7),$$

that is,
$$4\{6 + 1\} = 12 + 16.$$

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The formula (i) affords a verification of all values of $\sigma(n)$ with uneven arguments up to any limit and is very complete; but the multiplications required in the calculation of the terms are laborious. I used this formula to verify the portion from $n=1$ to $n=500$ of the table of $\sigma(n)$ communicated to the Society on January 28. The other three formulæ* are in no respect preferable to Euler's formula.

§ 5. Since the Meeting in January I have obtained the following curious formula which serves to express $\sigma(n)$ in terms of the σ's of *all* the numbers inferior to n, and thus affords a perfect and easy verification of a table of $\sigma(n)$.

If n be any number, then

$$\begin{aligned}
&\sigma(n)\\
&-2\sigma(n-1)-2\sigma(n-2)\\
&+3\sigma(n-3)+3\sigma(n-4)+3\sigma(n-5)\\
&-4\sigma(n-6)-4\sigma(n-7)-4\sigma(n-8)-4\sigma(n-9)\\
&+5\sigma(n-10)+\ldots\ldots\\
&\ldots\ldots\\
&\ldots\ldots+(-1)^{r-1}r\sigma(1)\\
&=(-1)^s\tfrac{1}{6}(s^3-s),
\end{aligned}$$

where s denotes what the coefficient of $\sigma(0)$ would be if the series were continued one term further. Thus $s=r$ unless the term $(-1)^{r-1}r\sigma(1)$ is the last of the group for which the coefficient is r, and when this is the case, $s=r+1$.

The expression on the right-hand side of the equation

$$=(-1)^s\tfrac{1}{6}(s-1)\,s\,(s+1)$$

and is thus obviously an integer. It will be noticed that the value of the series is the same for r consecutive positions of the last term, *i.e.* the value of the series is the same if the term involving $\sigma(1)$ is the last of the group having $r-1$ as coefficient or is any one except the last of the terms having r as coefficient.

For example, putting $n=5$, 6 and 7, we have

$$\begin{aligned}
&\sigma(5)\\
&-2\{\sigma(4)+\sigma(3)\}\\
&+3\{\sigma(2)+\sigma(1)\}\qquad =(-1)^3\tfrac{1}{6}(3^3-3),
\end{aligned}$$

* The formulæ (i) and (ii) were published in the *Quarterly Journal of Mathematics*, Vol. XIX. pp. 216, 222 (June, 1883). Since this paper was read the proofs of (iii) and (iv) have been published in the *Quarterly Journal*, Vol. XX. p. 118.

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$$\begin{aligned}
&\sigma(6)\\
&-2\{\sigma(5)+\sigma(4)\}\\
&+3\{\sigma(3)+\sigma(2)+\sigma(1)\}=(-1)^4\tfrac{1}{6}(4^3-4),
\end{aligned}$$

$$\begin{aligned}
&\sigma(7)\\
&-2\{\sigma(6)+\sigma(5)\}\\
&+3\{\sigma(4)+\sigma(3)+\sigma(2)\}\\
&-4\sigma(1) \qquad\qquad =(-1)^5\tfrac{1}{6}(4^3-4)
\end{aligned}$$

that is,

$$\begin{aligned}
6-2\{7+4\}+3\{3+1\} &= -4,\\
12-2\{6+7\}+3\{4+3+1\} &= 10,\\
8-2\{12+6\}+3\{7+4+3\}-4 &= 10.
\end{aligned}$$

The expression $\frac{1}{6}(s^3-s)$ is equal to the sum of the first $s-1$ triangular numbers so that the series is always numerically equal to the sum of the triangular numbers which do not exceed n.

If the series were continued one term further the next term would be $(-1)^{s-1}s\sigma(0)$, and if we put $\sigma(0)=\frac{1}{6}(s^2-1)$ this term becomes $(-1)^{s-1}\frac{1}{6}(s^3-s)$. We may therefore, by employing a convention of the same kind as Euler's, enunciate the theorem in the convenient form

$$\begin{aligned}
&\sigma(n)\\
&-2\sigma(n-1)-2\sigma(n-2)\\
&+3\sigma(n-3)+3\sigma(n-4)+3\sigma(n-5)\\
&-4\sigma(n-6)\ldots\ldots\\
&\ldots\ldots\\
&\ldots\ldots+(-1)^{s-1}s\sigma(0)\\
&=0,
\end{aligned}$$

where $\sigma(0)$ is defined to denote $\frac{1}{6}(s^2-1)$; s being the coefficient of $\sigma(0)$.

The table of $\sigma(n)$ up to $n=3000$, which was referred to at the end of the last section, will be verified by this formula during its passage through the press.

§ 6. I have obtained also the following formula which is of the same class. It does not however afford so complete a verification of a table as that given in the preceding section, as certain terms are omitted; and, further, as all the coefficients are ± 1 there is more chance of a compensation of errors.

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If P denote the expression

$$\begin{aligned} &\sigma(n) \\ &-\sigma(n-2)-\sigma(n-3)-\sigma(n-4) \\ &+\sigma(n-7)+\sigma(n-8)+\sigma(n-9)+\sigma(n-10)+\sigma(n-11) \\ &-\sigma(n-15)-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where between the groups of one, three, five, seven, ... terms having alternate signs there are gaps of one, two, three, ... terms, and the series is to be continued as long as the arguments remain positive, the term $\sigma(0)$ being included when it occurs but having the value zero; then, if $\sigma(0)$ occurs as a term,

$$P=(-1)^{r-1}\tfrac{3}{2}r(r+1),$$

where r denotes the number of terms omitted in the gap preceding the group of terms in which $\sigma(0)$ occurs, and, if $\sigma(0)$ does not occur as a term,

$$P=(-1)^{r}(r+1),$$

where r denotes the number of terms omitted in the last gap in the series.

For example, putting $n=7$, we have

$$\begin{aligned} &\sigma(7) \\ &-\sigma(5)-\sigma(4)-\sigma(3) \\ &+\sigma(0) \qquad\qquad =-\tfrac{3}{2}\times 2\times 3, \end{aligned}$$

that is, $8-6-7-4+0=-9.$

In this case r, the number of terms omitted between $\sigma(3)$ and $\sigma(0)$, $=2$.

Putting $n=8$, we have

$$\begin{aligned} &\sigma(8) \\ &-\sigma(6)-\sigma(5)-\sigma(4) \\ &+\sigma(1)+\sigma(0) \qquad\qquad =-\tfrac{3}{2}\times 2\times 3, \end{aligned}$$

that is, $15-12-6-7+1=-9,$

r being equal to 2 as before.

Putting $n=6$, we have

$$\begin{aligned} &\sigma(6) \\ &-\sigma(4)-\sigma(3)-\sigma(2) \;=-2, \end{aligned}$$

that is, $12-7-4-3=-2,$

for here, r, the number of terms omitted between $\sigma(6)$ and $\sigma(4)$, $=1$.

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Similarly,

$$\sigma(12) - \sigma(10) - \sigma(9) - \sigma(8) + \sigma(5) + \sigma(4) + \sigma(3) + \sigma(2) + \sigma(1) = 3,$$

that is, $$28 - 18 - 13 - 15 + 6 + 7 + 4 + 3 + 1 = 3,$$

the number of terms omitted between $\sigma(8)$ and $\sigma(5)$, being 2.

Thus, when $\sigma(0)$ occurs, $P = (-1)^{r-1} \frac{3}{2} r(r+1)$, and when $\sigma(0)$ falls in a gap and does not occur, $P = (-1)^r (r+1)$. In both cases r denotes the number of terms omitted in the last gap.

If we define $\sigma(0)$ to denote $\frac{1}{2}(r+1)(3r+2)$ where r is the number of terms omitted in the last gap we have, for all values of n,

$$P = (-)^r (r+1),$$

and this is perhaps the most convenient form in which to enunciate the theorem.

We may also express the theorem otherwise in the following forms:

(i) If $\sigma(0)$ occurs, and if s be the number of terms in the complete group to which it belongs, then $P = (-1)^{\frac{1}{2}(s+1)} \frac{3}{8}(s^2-1)$: if $\sigma(0)$ does not occur, and if t be the number of terms forming the gap in which it falls, then $P = (-1)^{t-1} t$.

(ii) If we define $\sigma(0)$ to denote n, and if A and B denote the numbers of positive and negative terms in the series, not counting $\sigma(0)$ as a term, then $P = A - B$.

Taking the same examples as before, we have, from (ii),

$$\sigma(7) - \sigma(5) - \sigma(4) - \sigma(3) + \sigma(0) = 1 - 3,$$

that is, $$8 - 6 - 7 - 4 + 7 = -2;$$

$$\sigma(8) - \sigma(6) - \sigma(5) - \sigma(4) + \sigma(1) + \sigma(0) = 2 - 3,$$

that is, $$15 - 12 - 6 - 7 + 1 + 8 = -1;$$

$$\sigma(6) - \sigma(4) - \sigma(3) - \sigma(2) = 1 - 3,$$

that is, $$12 - 7 - 4 - 3 = -2.$$

§ 7. I may also notice the following formula:

If n be any uneven number, then

$$n\sigma(n) + 2\{(n-2)\sigma(n-2) + (n-4)\sigma(n-4)\} + 3\{(n-6)\sigma(n-6) + (n-8)\sigma(n-8) + (n-10)\sigma(n-10)\} + 4\{(n-12)\sigma(n-12) + \ldots\ldots\}$$

$$\ldots\ldots\ldots

$$\begin{aligned}
&\sigma(n)\\
&+(1^2+3^2)\{\sigma(n-2)+\sigma(n-4)\}\\
&+(1^2+3^2+5^2)\{\sigma(n-6)+(n-8)+(n-10)\}\\
&+(1^2+3^2+5^2+7^2)\{\sigma(n-12)+\ldots\ldots\ldots\ldots\ldots\}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Both series are to be continued as far as the terms remain positive. As an example, putting $n=9$, we have

$$\begin{aligned}
&9\sigma(9)\\
&+2\{7\sigma(7)+5\sigma(5)\}\\
&+3\{3\sigma(3)+\sigma(1)\}.\\
&=\\
&\sigma(9)\\
&+10\{\sigma(7)+\sigma(5)\}\\
&+35\{\sigma(3)+\sigma(1)\};
\end{aligned}$$

that is,

$$9\times13+2\{56+30\}+3\{12+1\}=13+10\{8+6\}+35\{4+1\}.$$

§ 8. If we denote by $\zeta(n)$ the excess of the sum of the uneven divisors of n over the sum of the even divisors, it is evident that, if n be uneven, $\zeta(n)=\sigma(n)$; and it can be shewn that if n be even, and $=2^pr$, where r is uneven, then

$$\zeta(n)=-(2^{p+1}-3)\,\sigma(r).$$

This formula is true also when $p=0$ and includes the case of n uneven. It is evident that when n is even $\zeta(n)$ is negative.

The function ζ satisfies a relation of the same kind as Euler's formula, viz. if n be any number, we have

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\ldots=0, \text{ or } n,$$

according as n is not, or is, a triangular number.

If therefore we define $\zeta(n-n)=\zeta(0)$ to denote $-n$, we have, for all values of n,

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\ldots=0.$$

§ 9. The function $\zeta(n)$ may be expressed in terms of the ζ's of *all* the numbers inferior to n by means of the following formula, which differs from the σ-formula in § 5 only in the signs of the terms, which are all positive.

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If n be any number, then

$$\begin{aligned}
&\zeta(n)\\
&+2\zeta(n-1)+2\zeta(n-2)\\
&+3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5)\\
&+4\zeta(n-6)+\ldots\ldots\\
&\ldots\ldots\\
&\ldots\ldots+r\zeta(1)\\
&=\tfrac{1}{6}(s^3-s),
\end{aligned}$$

where, as in § 5, $s=r$, unless $r\sigma(1)$ is the last term of a group, in which case $s=r+1$.

As examples of the formula we find, putting $n=5$ and 6,

$$\begin{aligned}
&\zeta(5)\\
&+2\{\zeta(4)+\zeta(3)\}\\
&+3\{\zeta(2)+\zeta(1)\} &&=\tfrac{1}{6}(3^3-3),\\
&\zeta(6)\\
&+2\{\zeta(5)+\zeta(4)\}\\
&+3\{\zeta(3)+\zeta(2)+\zeta(1)\} &&=\tfrac{1}{6}(4^3-4);
\end{aligned}$$

that is,

$$\begin{aligned}
6+2\{-5+4\}+3\{-1+1\} &= 4,\\
-4+2\{6-5\}+3\{4-1+1\} &= 10.
\end{aligned}$$

If the series be continued one term further so as to include the term $s\zeta(0)$, and if we define $\zeta(0)$ to denote $-\frac{1}{6}(s^3-1)$, we may replace the right-hand member of the equation by zero.

§ 10. If we denote by $\Delta(n)$ the sum of the uneven divisors of n and by $D(n)$ the sum of the even divisors (so that $D(n)$ is zero when n is uneven), then

$$\begin{aligned}
\sigma(n)&=\Delta(n)+D(n),\\
\zeta(n)&=\Delta(n)-D(n),
\end{aligned}$$

and, by addition and subtraction from the formulæ in §§ 5 and 9, we find that, n being unrestricted,

$$\begin{aligned}
&\Delta(n)\\
&-2D(n-1)-2D(n-2)\\
&+3\Delta(n-3)+3\Delta(n-4)+3\Delta(n-5)\\
&-4D(n-6)-\ldots\ldots\\
&\ldots\ldots\\
&=0;
\end{aligned}$$

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$$\begin{aligned} &D(n) \\ &-2\Delta(n-1)+2\Delta(n-2) \\ &+3D(n-3)+3D(n-4)+3D(n-5) \\ &-4\Delta(n-6)-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &=0. \end{aligned}$$

The argument 0 is to be included in both formulæ, and we define $\Delta(0)$ to denote 0 and $sD(0)$ to denote $s \cdot \frac{1}{6}(s^2-1) = \frac{1}{6}(s^3-s)$.

For example, putting $n=5$ and 6, we have

$$\begin{aligned} &\Delta(5) \\ &-2D(4)-2D(3) \\ &+3\Delta(2)+3\Delta(1)+3\Delta(0)=0, \end{aligned}$$

that is, $6-12-0+3+3+0=0$;

$$\begin{aligned} &D(5) \\ &-2\Delta(4)-2\Delta(3) \\ &+3D(2)+3D(1)+3D(0)=0, \end{aligned}$$

that is, $0-2-8+6+0+\frac{1}{6}(3^3-3)=0$;

$$\begin{aligned} &\Delta(6) \\ &-2D(5)-2D(4) \\ &+3\Delta(3)+3\Delta(2)+3\Delta(1) \\ &-4D(0) \qquad\qquad =0, \end{aligned}$$

that is, $4-0-12+12+3+3-\frac{1}{6}(4^3-4)=0$;

$$\begin{aligned} &D(6) \\ &-2\Delta(5)-2\Delta(4) \\ &+3D(3)+3D(2)+3D(1) \\ &-4\Delta(0) \qquad\qquad =0, \end{aligned}$$

that is, $8-12-2+0+6+0-0=0$.

§ 11. The function which expresses the excess of the number of divisors of n which have the form $4m+1$ over the number of divisors which have the form $4m+3$ satisfies a relation so similar to the σ-formula of § 5 and the ζ-formula of § 9 as to be deserving of notice in connection with these two formulæ.

Denoting this function by $E(n)$, we find that, n being unrestricted,

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$$
\begin{gathered}
E(n) \\
-2E(n-1)+2E(n-2) \\
-3E(n-3)+3E(n-4)-3E(n-5) \\
-4E(n-6)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+(-)^{s-1}r\sigma(1) \\
=0 \text{ or } (-1)^{s-1}\tfrac{1}{2}s,
\end{gathered}
$$

according as s is uneven or even, s having the same meaning as in §§ 5 and 9.

§ 12. Corresponding to the ζ-formula in § 8 we have the formula

$$
\begin{gathered}
E(n)-E(n-1)+E(n-3)-E(n-6)+E(n-10)-\ldots \\
=0 \text{ or } (-1)^n\times\tfrac{1}{4}\{(-1)^{\frac{1}{2}\{\sqrt{(8n+1)}-1\}}\times\sqrt{(8n+1)}-1\},
\end{gathered}
$$

according as n is not, or is, a triangular number*.

§ 13. The three corresponding formulæ of §§ 5, 9 and 11 may be enunciated in a uniform manner as follows:

If n be any number, then

(i)

$$
\begin{gathered}
\sigma(n) \\
-2\sigma(n-1)-2\sigma(n-2) \\
+3\sigma(n-3)+3\sigma(n-4)+3\sigma(n-5) \\
-4\sigma(n-6)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+(-)^{s-1}s\sigma(0) \quad =0,
\end{gathered}
$$

where $\sigma(0)$ denotes $\frac{1}{6}(s^2-1)$.

(ii)

$$
\begin{gathered}
\zeta(n) \\
+2\zeta(n-1)+2\zeta(n-2) \\
+3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5) \\
+4\zeta(n-6)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+s\zeta(0)=0,
\end{gathered}
$$

where $\sigma(0)$ denotes $-\frac{1}{6}(s^2-1)$.

* The formulæ in §§ 11 and 12 are proved in a paper on the function $E(n)$ communicated to the London Mathematical Society on February 14, 1884.

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(iii)

$$E(n)$$
$$-2E(n-1)+2E(n-2)$$
$$-3E(n-3)+3E(n-4)-3E(n-5)$$
$$+4E(n-6)-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (-)^n sE(0) = 0,$$

where $E(0)$ denotes $\frac{1}{2}$ or 0 according as s is even or uneven.

In the σ-formula the groups of terms have alternately positive and negative signs, in the ζ-formula all the terms are positive, in the E-formula the signs of the terms are alternately positive and negative.

(2) *On primitive roots of prime numbers and their residues.* By A. R. FORSYTH, B.A.

(3) *A comparison of Maxwell's equations of the electromagnetic field with those of Helmholtz and Lorentz.* By R. T. GLAZEBROOK, M.A., F.R.S.

The equations of the Electromagnetic Field have been developed by Maxwell (*Electricity and Magnetism*, II.) on the one hand and by Helmholtz (*Borchardt's Journal*, Bd. LXXII. *Ueber die Bewegüngsgleichungen der Electricität*) and H. A. Lorentz (*Schlömilch Zeitschrift*, XXII.) on the other, starting from somewhat different standpoints. The object of the present communication is to give a comparative account of the two theories in the endeavour to discover the fundamental differences which lead to the different results actually arrived at.

According to Maxwell, when electromotive force acts on a medium, electric displacement is produced through it, the medium being polarized, and if P be the total electromotive force in the direction of the axis of x, the medium being isotropic, f the electric displacement and K the inductive capacity, we have $f = KP/4\pi$.

Suppose we take any section of the medium normal to the axis of x; then we know we can distribute electricity over that section so as to produce at any point of it the actual resultant electromotive force normal to the section, f, the electric displacement, will be the surface density of the electricity so distributed and to quote Maxwell (*Electricity and Magnetism*, Vol. I. § 62) "Whatever electricity be, and whatever we may understand by a

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movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a quantity of electricity through a wire is a movement of electricity."

Helmholtz considers the electric polarization set up in the medium by the electric force.

Each element of the medium becomes polarized, opposite electricities shewing themselves at the opposite ends.

Let us suppose that in the element of volume dv there are two quantities E and $-E$ of electricity at a distance s apart, these two quantities having each been moved the distance $\frac{1}{2}s$ by the action of the force. Then, according to Helmholtz, we may call Es the electric moment of the element, and the ratio of the electric moment to the volume of the element is the intensity of the electric polarization in the element in the direction of the resultant force. This intensity of the polarization can be resolved, and if ξ, η, ζ be its components then Helmholtz puts $\xi = \epsilon P$. P as before being the electromotive force and ϵ the dielectric constant of the medium.

To compare the two theories it becomes necessary then to determine the relation between f and ξ. The quantity ξ like f is a surface density, being the surface density of the electricity induced by the action of the electromotive force on the face of the element normal to Ox, while f as we have seen is the surface density of a distribution which will produce over the same surface the actual force.

Now according to Maxwell, if V be the potential at any point in the dielectric and ρ the density of the free electricity, then

$$K\nabla^2 V + 4\pi\rho = 0,$$

where

$$\nabla^2 = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2.$$

According to Helmholtz,

$$(1 + 4\pi\epsilon)\nabla^2 V + 4\pi\rho = 0.$$

The potential due to a quantity of electricity E placed at a point in the medium at distance r is on the two theories,

$$E/Kr \text{ and } E/(1 + 4\pi\epsilon)r.$$

Thus we see that

$$K = 1 + 4\pi\epsilon,$$

and

$$f = \frac{K}{4\pi\epsilon}\xi = \frac{1 + 4\pi\epsilon}{4\pi\epsilon}\xi.$$

We may compare these equations with those in Poisson's theory of induced magnetization (Maxwell, Vol. II. § 426, etc.).

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Calling κ the coefficient of induced magnetization, A, B, C the components of magnetization, α, β, γ components of magnetic force and a, b, c of magnetic induction, we have

$$A = \kappa\alpha, \text{ etc.},$$

$$a = \alpha + 4\pi A = \alpha(1 + 4\pi\kappa) = \frac{A(1 + 4\pi\kappa)}{\kappa}.$$

Thus ξ, η, ζ correspond to A, B, C; $4\pi f$, $4\pi g$, and $4\pi h$ to a, b, c respectively, so that if we call $4\pi f$, etc. the components of the electric induction, we may say that Maxwell's equations deal with the electric induction, Helmholtz's with the electric polarization produced by the given electromotive force. Moreover μ the magnetic inductive capacity of the substance is related to κ the coefficient of induced magnetization in the same way that K the electric inductive capacity is related to ϵ the coefficient of induced electric polarization, for we have

$$\mu = 1 + 4\pi\kappa,$$

and

$$K = 1 + 4\pi\epsilon.$$

With this understood then it is clear that the two equations

$$\left.\begin{aligned} f &= KP/4\pi \\ \xi &= \epsilon P \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

are perfectly consistent.

The value of P depends in part on the electromagnetic action of the medium. Now Helmholtz uses electrostatic units throughout and therefore has to multiply the terms in P which arise from the electromagnetic action by a constant A^2 such that A expresses the number of electromagnetic units in one electrostatic unit of current. If we suppose throughout that our quantities are measured in electromagnetic units we may put $A =$ unity; we shall do this throughout.

We have now to consider the electromagnetic effects produced by the currents and magnets in the field.

Helmholtz treats the two separately; according to him the effects of the currents depend on three quantities U, V, W satisfying certain equations which will be referred to again shortly, while the effects of the permanent magnets depend on three other quantities L, M, N to be further defined in the sequel.

According to Maxwell, on the other hand, the electromagnetic effects at any point depend on the electro-kinetic momentum at this point, and this we may write

$$F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}.$$

We may, in order to compare this with Helmholtz, also divide F, G, H into two parts F_1 and F_2, etc., where F_1, etc. arise from

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the action of currents, F_2, etc. from the direct action of magnetic bodies. If we adopt Ampère's hypothesis that magnetism is due to molecular electric currents, the terms in F_2, etc. disappear.

The electromotive force at the point parallel to the axis of x is $-\frac{dF}{dt}$, which is equal to $-\frac{dF_1}{dt}-\frac{dF_2}{dt}$. According to Helmholtz the part of the electromotive force which arises from the action of the currents is $-\mu\frac{dU}{dt}$, so that

$$\frac{dF_1}{dt}=\mu\frac{dU}{dt}, \text{ etc.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, let F_2 arise from magnetism distributed throughout space in such a way that its components at x', y', z' are A, B, C, let p be the reciprocal of the distance between two points whose co-ordinates are x, y, z and x', y', z'. Then (Maxwell, II. § 405)

$$F_2=\iiint\left(B\frac{dp}{dz'}-C\frac{dp}{dy'}\right)dx'dy'dz', \text{ etc.}$$

Hence if we put $L=\iiint Apdx'dy'dz'$, $M=$ etc., $N=$ etc.,

we have
$$F_2=\frac{dM}{dz}-\frac{dN}{dy},$$

and
$$-\frac{dF_2}{dt}=\frac{d}{dt}\left\{\frac{dN}{dy}-\frac{dM}{dz}\right\}.$$

This agrees with Helmholtz's expression just referred to for the part of the E. M. F. which depends on magnets.

Again, to find the magnetic force due to this let α, β, γ be the components of the magnetic force, a, b, c of the magnetic induction and χ the magnetic potential.

Then, we have

$$\chi=-\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)$$

and
$$\begin{aligned}\mu\alpha=a&=\frac{dH}{dy}-\frac{dG}{dz}\\&=\frac{dH_1}{dy}-\frac{dG_1}{dz}+\frac{dH_2}{dy}-\frac{dG_2}{dz}\\&=\frac{dH_1}{dy}-\frac{dG_1}{dz}+\frac{d}{dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)-\nabla^2L\\&=\frac{dH_1}{dy}-\frac{dG_1}{dz}-\frac{d\chi}{dx}+4\pi A\\&=\frac{dH_1}{dy}-\frac{dG_1}{dz}+\mu\alpha_2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),\\&\quad\text{etc.}\ldots\ldots\ldots\ldots\end{aligned}$$

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if α_2 denote the magnetic force due to the magnetization. According to Helmholtz the magnetic force is expressed by

$$\frac{dV}{dz} - \frac{dW}{dy} + \alpha_2,$$

α_2 being as above the part due to magnetization and U, V, W the quantities already referred to, and these are consistent with the equations $F_1 = \mu U$, etc. derived from (2) by integrating and omitting a function of the form $\frac{d\lambda}{dx}$ where λ is independent of the time.

Moreover α_2, etc. being derivable from a potential function, we have

$$\mu\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) = \frac{d}{dx}\left(\frac{dF_1}{dx} + \frac{dG_1}{dy} + \frac{dH_1}{dz}\right) - \nabla^2 F_1$$

$$= \frac{dJ_1}{dx} - \nabla^2 F_1 \ldots\ldots\ldots(4),$$

if $$J_1 = \frac{dF_1}{dx} + \frac{dG_1}{dy} + \frac{dH_1}{dz}.$$

But according to Maxwell, *Electricity and Magnetism*, II. 616,

$$\mu\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) = \frac{dJ}{dx} - \nabla^2 F \ldots\ldots\ldots\ldots\ldots(5),$$

with two similar equations if

$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}.$$

Also from the values of F_2, G_2, H_2 it follows that

$$\frac{dF_2}{dx} + \frac{dG_2}{dy} + \frac{dH_2}{dz} = 0,$$

$$\therefore \quad J = J_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Hence from (4), (5) and (6) we have

$$\nabla^2 F = \nabla^2 F_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Thus on either theory we have satisfied the three equations of which the type is

$$\mu\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) = \frac{dJ_1}{dx} - \nabla^2 F_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or introducing the U, V, W of Helmholtz's notation

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = \frac{d}{dx}\left(\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz}\right) - \nabla^2 U$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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Moreover the expression for the electromotive force which arises from electromagnetic induction is the same on the two theories, being according to Maxwell given by equations of the type

$$P = -\frac{dF_1}{dt} + \frac{d}{dt}\left(\frac{dN}{dy} - \frac{dM}{dz}\right) \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and according to Helmholtz in a magnetizable medium

$$P = -\mu\frac{dU}{dt} + \frac{d}{dt}\left(\frac{dN}{dy} - \frac{dM}{dz}\right) \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

The two equations are identical and may be written

$$P = -\frac{dF}{dt} \ldots\ldots$$

Before however we can solve the problem completely we must know the relation which exists between the values of F, G, and H, or U, V, W and the current at any distant point.

Let u, v, w be the components of the total electric current at any point. Then since F, G, H depend on the action of the distant current it is clear that there must be some relation between them and the values of u, v, w. If we are considering the whole electro-kinetic momentum round a closed curve we know the form of the equation which expresses the connexion—for if i be the current at any point distant r from the point considered and ϵ the angle between $d\sigma$ an element in the direction of i and that of an element ds of this closed curve, then

$$\int F_1 dx + G_1 dy + H_1 dz = \mu \int \frac{i\cos\epsilon}{r}\, ds\, d\sigma \ldots$$

Helmholtz starts from this and by means of some transformations arrives at the equations

$$\nabla^2 U = (1-k)\frac{d^2\Phi}{dx\,dt} - 4\pi u \ldots$$

(or in our notation) etc.

$$\nabla^2 F_1 = \mu(1-k)\frac{d^2\Phi}{dx\,dt} - 4\pi\mu u \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

etc.,

k being a certain constant and $\dfrac{d\Phi}{dt}$ defined by the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} - \frac{1}{4\pi}\nabla^2\frac{d\Phi}{dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

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From (10) and (11) we have by differentiation

$$J = J_1 = \frac{dF_1}{dx} + \frac{dG_1}{dy} + \frac{dH_1}{dz} = -\mu k \frac{d\Phi}{dt} \quad \text{.....(12),}$$

an equation which can also be obtained directly from the values of F_1, G_1, H_1 found by Helmholtz; and since

$$J = J_1, \quad \nabla^2 F = \nabla^2 F_1,$$

Helmholtz's fundamental equations may be written

$$\nabla^2 F + \frac{1-k}{k} \frac{dJ}{dx} = -4\pi\mu u \quad \text{.....(13),}$$

etc.

$$\therefore \quad \nabla^2 F - \frac{dJ}{dx} = -4\pi\mu u - \frac{1}{k} \frac{dJ}{dx} \quad \text{.....(14).}$$

etc.

Hence from (5)

$$\mu \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) = 4\pi\mu u + \frac{1}{k} \frac{dJ}{dx} = 4\pi\mu u - \mu \frac{d^2\Phi}{dxdt} \quad \text{.....(15).}$$

etc.

Also differentiating these with reference to x, y, z and adding

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + \frac{1}{4\pi\mu k} \nabla^2 J = 0 \quad \text{.....(16).}$$

Maxwell's equations differ from these. According to him we have instead of (15)

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u \quad \text{.....(17).}$$

Hence $$\frac{d^2\Phi}{dxdt} = 0, \text{ etc.,}$$

and therefore $$\frac{J}{k} = 0.$$

Hence either $$J = 0 \text{ or } k = \infty.$$

If J is not to be zero then to make Helmholtz's theory coincide with Maxwell's we must put $k = \infty$ and $\frac{d\Phi}{dt} = 0$. According to Helmholtz the two agree if $k = 0$. We shall have to return to this point again.

In order to proceed further we require to evaluate P, Q, R the components of the electric force in terms of the other quantities, and here a further difference between the two occurs. The value

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of P will depend partly on electromagnetic, partly on electrostatic forces, and partly on the polarization of the dielectric produced by these. In both theories we write $-\frac{dF}{dt}$ for the first part. Helmholtz and Lorentz denote the electrostatic part by X, the part depending on polarization by $-\frac{d\Phi}{dx}$. Maxwell, assuming that the electrostatic forces have a potential, combines the two into one term $-\frac{d\Psi}{dx}$. In cases in which the assumption is true the two methods should lead to the same result.

We shall however for the sake of convenience in the future work put Ω and not Φ for the potential of the forces produced by the polarization and write

$$P = -\frac{dF}{dt} - \frac{d\Omega}{dx} + X \quad \text{.....(18),}$$

remembering that according to Helmholtz $\Omega = \Phi$, while Maxwell treating only the case in which X, Y, Z are derivable from a potential puts generally,

$$\frac{d\Omega}{dx} - X = \frac{d\Psi}{dx}.$$

Then we may state that the values of P, Q, R in the right-hand sides of the two sets of equations

$$\xi = \epsilon P, \text{ etc.},$$

$$f = \frac{K}{4\pi} P, \text{ etc.},$$

are the same in form though the equations which connect F, G, H with the components of the current are different.

And this brings us to another distinct point of difference. According to Maxwell the components of the current in a dielectric are $\dot{f}$, $\dot{g}$, $\dot{h}$, according to Helmholtz they are $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$.

Hence taking Helmholtz's supposition first, we have from (15), (1) and (18),

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} + \frac{d^2\Phi}{dx\,dt} = 4\pi\dot{\xi} = 4\pi\epsilon \frac{dP}{dt}$$

$$= -4\pi\epsilon \frac{d}{dt}\left(\frac{dF}{dt} + \frac{d\Omega}{dx} - X\right) \quad \text{.....(19),}$$

$$\text{etc.....}$$

and two similar equations.

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From these we obtain by some simple well-known transformations,

$$\frac{d^2\alpha}{dt^2} = \frac{1}{4\pi\mu\epsilon}\nabla^2\alpha \quad \text{.....(20).}$$

etc.

Thus the magnetic force is propagated as a wave with velocity

$$\sqrt{\frac{1}{4\pi\mu\epsilon}},$$

and it can be shewn that the direction of the force is in the plane of the wave.

This is Helmholtz's result.

According to Maxwell we have $\dot{f}$, $\dot{g}$, $\dot{h}$ as the components of the current, so that instead of (19) we obtain

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} + \frac{d^2\Phi}{dx\,dt} = 4\pi\dot{f}$$

$$= K\frac{dP}{dt} = -K\frac{d}{dt}\left(\frac{dF}{dt} + \frac{d\Omega}{dx} - X\right) \text{.....(21),}$$

and equations (20) become

$$\frac{d^2\alpha}{dt^2} = \frac{1}{\mu K}\nabla^2\alpha \text{.....(22),}$$

so that the velocity of propagation of the disturbance is $\sqrt{\left(\frac{1}{\mu K}\right)}$ instead of $\sqrt{\frac{1}{4\pi\mu\epsilon}}$.

This is the result which follows from Helmholtz's theory on Maxwell's supposition as to the value of the current.

The different expressions obtained on the two theories for the velocity arise solely from the different assumptions made as to what is a current in a dielectric, and are independent of the value of the unknown constant k. Again, keeping to Helmholtz's assumption as to the current, we have

$$\nabla^2 F + \frac{1-k}{k}\frac{dJ}{dx} = -4\pi\mu\dot{\xi}$$

$$= 4\pi\mu\epsilon\left\{\frac{d^2F}{dt^2} + \frac{d^2\Omega}{dx\,dt} - \frac{dX}{dt}\right\} \text{.....(23).}$$

Differentiate with reference to x, y, and z and add, then

$$\nabla^2 J = 4\pi\mu k\epsilon\left[\frac{d^2J}{dt^2} + \frac{d}{dt}\left\{\nabla^2\Omega - \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right)\right\}\right] \text{.....(24).}$$

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Now it is from this equation modified by the suppositions that $\Omega = \Phi$, and that $\frac{d}{dt}\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) = 0$, that Helmholtz obtains his normal wave.

For the equation expressed in terms of J then becomes

$$4\pi\mu k\epsilon \frac{d^2J}{dt^2} = (1 + 4\pi\epsilon)\nabla^2 J \ldots\ldots\ldots\ldots(25).$$

Hence J and therefore Φ, since $J = -\mu k \frac{d\Phi}{dt}$, travel with a velocity

$$\sqrt{\frac{(1 + 4\pi\epsilon)}{4\pi\mu k\epsilon}}.$$

This agrees with Helmholtz if we remember that in a magnetic medium of magnetic inductive capacity μ he replaces k by k/μ. Thus the existence of a longitudinal wave travelling with this velocity depends on the fact that Helmholtz assumes that Φ is the potential arising in the dielectric from the polarization; if we do not make this supposition then equation (24) gives us, omitting as before, the terms in X, Y and Z,

$$\frac{d}{dt}\nabla^2\Omega = \frac{1}{4\pi\mu k\epsilon}\nabla^2 J - \frac{d^2J}{dt^2} \ldots\ldots\ldots\ldots\ldots\ldots(26),$$

or

$$\nabla^2\Omega = \mu k \frac{d^2\Phi}{dt^2} - \frac{1}{4\pi\epsilon}\nabla^2\Phi \ldots\ldots\ldots\ldots\ldots(27).$$

If we adopt Maxwell's view as to the current we must distinguish thus between Φ and Ω, for Φ is then the potential due to matter of density $-\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)$, while Ω is that due to matter of density $-\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)$.

We arrive at Maxwell's case by putting

$$\Phi = 0 \text{ and } -\mu k \frac{d\Phi}{dt} = J,$$

and then the relation between Ω and J becomes

$$\frac{d}{dt}\nabla^2\Omega = -\frac{d^2J}{dt^2} \ldots\ldots\ldots\ldots\ldots\ldots(28).$$

This corresponds to Maxwell's equation 8 (*Electricity and Magnetism*, II. 783), for since $\frac{dX}{dx} + \ldots = 0$,

$$\nabla^2\Omega = \nabla^2\Psi^* \ldots\ldots\ldots\ldots\ldots\ldots(29).$$

* In Maxwell ∇^2 has the opposite sign to that with which it is used here.

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Now Maxwell concludes that $\nabla^2\Psi$ is proportional to the volume density of free electricity. This is of course the case if there is no electromagnetic induction, but when it exists we have (ρ being the volume density) on his theory

$$\rho = \frac{df}{dx} + \frac{dh}{dy} + \frac{dg}{dz} = -\frac{K}{4\pi}\left(\nabla^2\Omega + \frac{dJ}{dt}\right) \quad \text{.....(30).}$$

and if we suppose with Maxwell that, the medium being an insulator, $\frac{d\rho}{dt} = 0$ we get

$$\frac{d}{dt}\nabla^2\Omega = -\frac{d^2J}{dt^2},$$

exactly the same equation as that which flows from Helmholtz's theory by putting $\Phi = 0$ and $-\mu k\frac{d\Phi}{dt} = J$.

We have already noticed that Helmholtz states that his theory reduces to Maxwell's by putting $k = 0$. He arrives at this result by considering the normal wave which does not exist on Maxwell's theory, and states that according to Maxwell its velocity is infinite. The comparison between the two theories shews us that they will be reconciled if we put $\frac{d\Phi}{dt} = 0$ everywhere. If it be a part of Maxwell's theory that J should not be zero, then since $J = -\mu k\frac{d\Phi}{dt}$ we must have k infinite and not zero, and we obtain Maxwell's relation between J and Ω; while the normal wave of Helmholtz' theory disappears, its velocity becoming zero.

If however we assume that $J = 0$ is an essential condition of Maxwell's theory the two are reconciled independently of the value of k by the supposition that $\frac{d\Phi}{dt} = 0$ and the normal wave depending on Φ disappears.

[*Note added August*, 1884.]

Since the above was in type Mr J. J. Thomson has pointed out to me that the supposition $k = \infty$ will make the action, on Helmholtz' theory, of one element of current on another infinite.

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For this action depends on the value of

$$\frac{1}{2r}[\cos(ds, d\sigma) + \cos(r, ds)\cos(r, d\sigma) + k\{\cos(ds, d\sigma) - \cos(r, ds)\cos(r, d\sigma)\}],$$

$ds, d\sigma$ being the elements of current and r the distance between them, and if k is infinite this expression is infinite also, though by integrating throughout a space whose boundaries are infinitely distant from the point considered it leads us to

$$F_1 = \mu \iiint \left(-\frac{1-k}{4\pi} \frac{d^2\Phi}{dt\,dx'} + u \right) \frac{1}{r}\, dx'dy'dz',$$

which on putting $\frac{d\Phi}{dt} = 0$ and $-\mu k \frac{d\Phi}{dt} = J$ gives

$$F_1 = -\frac{1}{4\pi} \iiint \frac{dJ}{dx'} \frac{1}{r}\, dx'dy'dz' + \mu \iiint \frac{u}{r}\, dx'dy'dz',$$

so that F_1 is finite.

Since however we cannot adopt an infinite value for the action of one element on another we are forced to conclude that to reconcile Maxwell's theory with that of Helmholtz we must have $J = 0$, and this condition will be satisfied, for any value of k which is not infinite, whenever $\Phi = 0$; thus the assumption $\Phi = 0$ is sufficient for our purpose independently of the value of k provided at least it be not infinite.

In this case from (26) we have $\frac{d}{dt}(\nabla^2\Omega) = 0$, and comparing this with (30) we get $\frac{d\rho}{dt} = 0$, so that no free electricity is produced in the medium by the electromagnetic induction; there will be no normal wave, and the quantity k disappears from our equations.

From a physical point of view this condition $\Phi = 0$, at least in a dielectric, seems the reasonable one. For Φ, according to Helmholtz, is the potential of the electrification produced throughout the dielectric by the given electromotive forces. So far as we know it is not possible to generate electricity by induction in the interior of a dielectric; the distribution produced will, like that in the similar magnetic problem, be solenoidal, and the density of the electricity which gives rise to the potential Φ will therefore be zero everywhere, and Φ will be zero.

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March 10, 1884.

Mr GLAISHER, PRESIDENT, IN THE CHAIR.

D'Arcy W. Thompson, B.A., Trinity College, was elected a Fellow.

The following communications were made:

(1) *Continuation of Observations on the state of an Eye affected with Astigmatism.* By Sir GEORGE BIDDELL AIRY, K.C.B., M.A., LL.D., D.C.L., Honorary Fellow of Trinity College, Cambridge; formerly Lucasian Professor and Plumian Professor in the University of Cambridge; late Astronomer Royal.

In making and treating the observations now presented to the Cambridge Philosophical Society, I have used exactly the same simple method which I have employed on four previous occasions, and of which the details are preserved in the *Transactions* of the Society. Without entering into further particulars, I now present the immediate results of these observations, combined with those of preceding examinations, in the same form as those which are printed in Volume XII. Part I. of the *Transactions*. I have however added the column of personal age, as bearing in some measure on the explanation of the changes.

I. Distance from the cornea of the left eye, at which a luminous point presents the appearance of a nearly horizontal line.

In 1825, age 24, distance 3·5 inches; Reciprocal = ·286.

Difference −·073

In 1846, ... 45, ... 4·7 .....; ..... ·213

..... −·028

In 1866, ... 65, ... 5·4 .....; ..... ·185

..... −·006

In 1871, ... 70, ... 5·6 .....; ..... ·179

..... −·011

In 1884, ... 83, ... 5·95 .....; ..... ·168

II. Distance from the cornea of the left eye, at which a luminous point presents the appearance of a nearly vertical line.

In 1825, age 24, distance 6·0 inches; Reciprocal ·166

Difference −·054

In 1846, ... 45, ... 8·9 .....; ..... ·112

..... −·018

In 1866, ... 65, ... 10·6 .....; ..... ·094

..... +·006

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In 1871, age 70, distance 10·0 inches; Reciprocal ·100

Difference —·029

In 1884, ... 83, ... 14·1 .....; ..... ·071

III. Measures of the astigmatic power of the left eye at different epochs, estimated in each case by the differences of the reciprocals of the same date in the two preceding tables.

In 1825, age 24, astigmatism = ·120

Difference = — ·019

In 1846, ... 45; ..... ·101

..... — ·010

In 1866, ... 65, ... ·091

..... — ·012

In 1871, ... 70, ... ·079

..... + 0·18

In 1884, ... 83, ... ·097

It appears probable that the measures in the second table are somewhat erroneous, perhaps more particularly that for 1871. It is not easy to make the observations at the greater distance.

I am unable to give measures for the right eye corresponding to those of preceding years. A defect, which has now existed for many months, totally destroys vision in the centre of the field of view.

(2) *On the measurement of the electrical resistance between two neighbouring points on a conductor.* By Lord RAYLEIGH, M.A. (With an account of experiments by R. W. SHACKLE, M.A., and A. W. WARD, B.A.)

As an alternative to the method of Matthiessen and Hockin, the following process may be used with good effect for the above purpose. It is founded upon the combination of resistances described in a paper upon the determination of the ohm by Lorenz's method (*Phil. Trans.* 1883), by which a small *effective* resistance is obtained from elements of moderate and accurately measurable value.

The main current γ passes principally through a shunt of resistance a, but partly through a small resistance b and a large resistance c arranged in series. The terminals of a galvanometer of resistance g are connected to the extremities of b. If g were infinite, the difference of potentials at its electrodes would be

$$\frac{ab}{a+b+c}\gamma,$$

so that $ab/(a+b+c)$ is the effective resistance of the combination. For example, if $a=1$, $b=1$, $c=98$, the effective resistance is $\frac{1}{100}$,

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and notwithstanding its smallness is susceptible of accurate determination. Suppose now that the main current traverses also a German silver strip (*Proc.*, Nov. 26, 1883) provided with tongues between which we require to know the resistance. It is evident that by adjustment of c the combination may be made to give the same effect upon the galvanometer as the German silver strip, so that the required result would be readily obtained from the above formula. If c is taken from a resistance-box, we may find the effects, one greater and one less than that of the strip, corresponding to resistances c_0 and c_0+1, whence the value that would give exactly the same effect is deduced by interpolation. In order to guard against disturbance from thermo-electricity the readings should be taken by *reversal* of the battery, and to eliminate the effects of varying current the combination and the strip should be interchanged as rapidly as possible.

In practice the resistance of the galvanometer could not usually be treated as infinite, and the interpretation of the results is a little more complicated. In the case of the combination it may be shewn that the current through the galvanometer is

$$\frac{ab\gamma}{g(a+b+c)+b(a+c)}.$$

By putting a infinite, or otherwise, we see that the corresponding current for the strip is $x\gamma(g+x)$, if x be the required resistance between the tongues. Equating these, we find

$$x=\frac{abg}{g(a+b+c)+bc}.$$

This method has recently been tested in the Cavendish Laboratory by Messrs Shackle and Ward (see below), and the results appear to shew that even with so moderate a main current as ·2 ampère, the sensitiveness is sufficient, the mean of a few readings being probably correct to $\frac{1}{1000}$.

Details of the experiment by Messrs Shackle and Ward.

The arrangement of the apparatus is given in the subjoined diagram.

The battery used was a Daniell's, an extra resistance of 2 ohms being inserted in the circuit. The battery terminals were connected by a reversing key as shewn in the diagram. A, B, C, D are mercury cups with the terminals of the standard ohms a and b firmly pressed into them: B and C are connected by a thick

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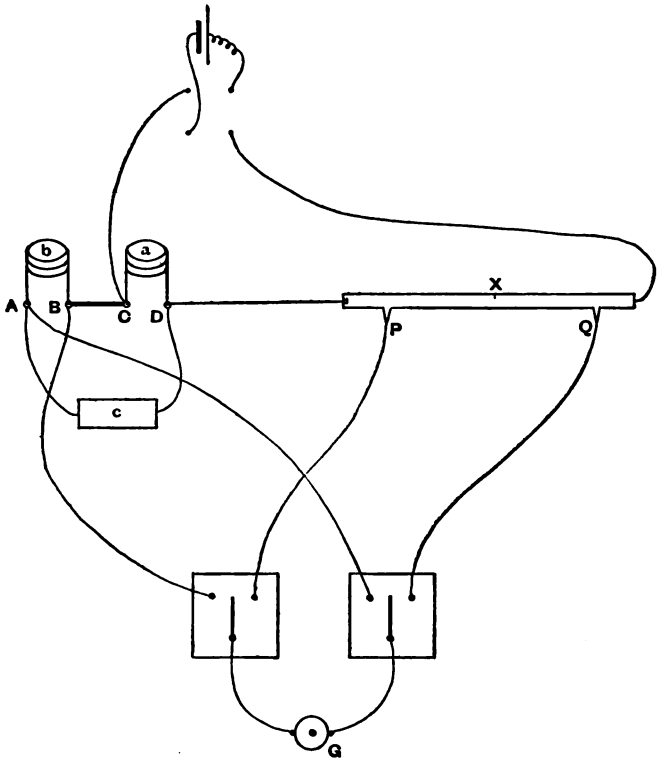
copper wire. c is a resistance box of 1—200 ohms. The switch keys connected with the galvanometer terminals enable us to change the galvanometer circuit from ABG to PQG with as little delay as possible.



The readings which were taken are given in the following table. The 1st column gives the resistance in the box c: the 2nd column gives the deflection of galvanometer to the left when the circuit P, Q, G is completed, and the 3rd column the corresponding deflection to the right when the current is reversed. The 5th and 6th columns give the readings in similar order when the circuit ABG is completed. To eliminate the slight variations in the electromotive force of the battery the order of observations adopted in any row was that recorded in 2nd, 5th, 3rd, 6th columns, or 5th, 2nd, 3rd, 6th columns: this made the total deflections correspond as nearly as possible to the same instant of time.

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	Current through *PQG*.				Current through *ABG*.			
	c.	Defl. to left.	Defl. to right.	Total.	Defl. to left.	Defl. to right.	Total.	Differences.
	182	180·5	221·4	401·9	182·5	221·7	404·2	
	„	181·5	222·4	403·9	183·5	223·4	406·9	
	„	182	224	406	184	224·7	408·7	
Mean				403·93			406·6	2·67
	183	182	223	405	183	223	406	
	„	183	223·3	406·3	182·5	224·5	407	
	„	183	224·7	407·7	183·5	224·5	408	
Mean				406·33			407	·67
	184	183·5	225·3	408·8	183	225	408	
	„	183·2	228	411·2	182·5	227·7	410·2	
Mean				410			409·1	− ·9
	185	183	226·7	409·7	181·3	225·5	406·8	
	„	182·7	226·5	409·2	181·3	225	406·3	
	„	183·7	226·7	410·4	181·5	225	406·5	
Mean				409·6			406·53	− 3·23

If we interpolate between $c = 182$ and $c = 185$ we get for the true value of c 183·36, and using the values $c = 183$, and 184, we get for true value $c = 183{\cdot}43$. Substituting these values of c in the value of x, and putting $a = 1$, $b = 1$, $G = 65{\cdot}6$ as determined by direct measurement, we get

$$x = {\cdot}005314 \text{ for } c = 183{\cdot}36$$

and

$$= {\cdot}005313 \text{ for } c = 183{\cdot}43.$$

The temperature was 14° C.

To find limits of error.

We have

$$x = \frac{abG}{G(c + a + b) + bc},$$

and to a first approximation, $a = 1$, $b = 1$, $G = 65$, $c = 183$, $x = \frac{1}{200}$. Hence

$$\frac{dx}{x} = da(1 - x) + db\left(1 - x\frac{c + G}{G}\right) - dcx\frac{G + 1}{G} + \frac{dG(1 - cx)}{G}$$

$$= da + db - \frac{dc}{200} + \frac{dG}{700}.$$

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In the present case da and db are known to be less than ·0005, and dG is less than ·1. The errors of c are due to inaccuracy in resistance box and to errors in reading the galvanometer deflections. The first we may dismiss as less than ·01, the second we proceed to find.

Let us call the differences given in last column of table, h (= ·67 app.) when $c = 183$, and k (= ·9) when $c = 184$. By interpolation the true value of c is

$$183 + \frac{h}{h+k},$$

$$\therefore dc = \frac{kdh - hdk}{(h+k)^2}.$$

Since dh and dk are certainly not greater than ·3 the greatest possible value of dc is $\frac{\cdot 3}{h+k} = \frac{1}{5}$. This gives $\frac{dc}{200} < \frac{1}{1000}$.

We may then conclude that the accuracy of the determination is 1 in 1000.

This result has been verified by using Matthiessen and Hockin's method, when more than six independent experiments have given the same result as above to the 3rd significant figure.

The experiment just described was not made in the best way possible. Having once found the limits 183 and 184 between which c lay, it would have been best to have restricted the observations entirely to those limits, and by carefully adjusting the galvanometer and scale, the errors dh and dk might no doubt be reduced. It would not be too much to say that the necessary accuracy could have been carried to the 4th significant figure, though this would involve some careful attention to changes of temperature.

(3) *On dimensional equations and change of units.* By W. N. Shaw, M.A.

Since the introduction of methods of measuring electrical and magnetic quantities in absolute measure considerable attention has necessarily been turned to the question of the dimensions of units and dimensional equations. Maxwell, as is well known, has in various places discussed such questions, and they naturally form an important part of Everett's 'Units and Physical Constants.' But I do not recollect having anywhere seen any precise statement of the manner in which dimensional equations arise and what their actual significance is. I therefore venture to suggest the following exposition of the method of deducing dimensional equations, and I do so with more confidence as there seems a general tendency to attribute to the well-known symbol in square brackets more of the

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attributes of an actual concrete quantity than it is justly entitled to.

We may accept in the first place, as usual, that the complete expression of a physical quantity consists of two parts and may be represented by the symbol $q\ [Q]$, where q represents the numerical part of the expression and $[Q]$ the *concrete unit of its own kind* which has been selected for the measurement of the quantity.

The unit $[Q]$ is initially arbitrary for every kind of quantity. There exist however certain quantitative physical laws which really express by means of variation equations relations between the numerical measures of quantities. We may take for instance the following to be the expression of Ohm's law: "The numerical measure of the current in an elongated conductor varies directly as the electromotive force between the ends of the conductor." Or Oersted's discovery may be summed up as follows: "The numerical value of the force upon a magnetic pole placed at the centre of a circular arc of wire conveying a current, varies directly as the strength of the pole, as the length of the wire, as the strength of the current, and inversely as the square of the radius of the arc." A very large number of similar instances might be given.

We may thus take as the expression of a physical law the general form

$$q \propto x^{\alpha} y^{\beta} z^{\gamma} \ldots$$

where $q, x, y, z \ldots$ are the *numerical measures* of the different quantities concerned in the relation.

We may of course express the variation equation in the form

$$q = k\, x^{\alpha} y^{\beta} z^{\gamma} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where k is some constant whose value in general alters, if we alter the units in which the different quantities are measured, for by so doing we alter in the inverse ratio the numerical values $x, y, z \ldots$

We may adopt one of two courses with respect to the quantity k.

(1) If all the possible variables have not been accounted for we may regard k as a fresh variable. This has been done in the instance first quoted, viz. in that of Ohm's Law, where k depends on the nature of the conductor. Thus the reciprocal of k in that instance is now generally known as the 'resistance' of the conductor, and we re-state the law thus: "The current in the elongated conductor varies directly as the electromotive force between the ends and inversely as the resistance of the conductor," and the expression of the law becomes

$$c = k' \frac{e}{r}.$$

So that we are still left with an equation of similar form, and

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hence may regard the equation (1) in the final general form of the expression of any physical law.

(2) We have already mentioned that the numerical value of k depends upon the units $[Q]$, $[X]$, $[Y]$, $[Z]$... employed to measure the different quantities. We may therefore assign to k any value we please by a suitable choice of any one of the units $[Q]$, $[X]$, $[Y]$...

For many reasons it is convenient that k should be unity, and therefore the most usual assumption is that the unit of $[Q]$ should be so chosen that k shall be unity.

In the same manner x, y and z can generally be connected by physical laws with the three units of mass, space and time, and we may thus obtain $k = 1$ for a large number of physical equations, provided the whole series of units are chosen on the principle here indicated. We thus see that systems of units can be formed based on three fundamental units, such that a whole series of physical laws, expressing relations between the quantities measured, can be represented by ordinary equations with constant unity, instead of by variation equations. We thus arrive at systems of units founded on this principle, and a unit belonging to such a system is called an absolute unit. For such a unit the right of arbitrary choice has been given up, and it is agreed that the choice shall be directed by a consideration that the quantity k in certain equations shall be made equal to unity.

It follows from this that when the three fundamental units are selected the rest of the units belonging to the system are thereby defined, and that if the fundamental units are altered, corresponding alterations must take place in the whole system based upon the three fundamental units, in order that the k's may be still maintained equal to unity.

Let us consider the change from one system of absolute units to another, both founded upon the same principle, that is to say, both agreeing that the same k's shall be unity.

The equation between the *numerical measures* of q, x, y, z... thus becomes for *both* systems

$$q = x^{\alpha}\ y^{\beta}\ z^{\gamma} \ldots .$$

Let the unit of x be changed from $[X]$ to $[X']$,
..... y ..... $[Y]$ ... $[Y']$,
..... z ..... $[Z]$ ... $[Z']$,

and in consequence

q ..... $[Q]$ ... $[Q']$.

Then if q', x', y', z' be the new numerical measures of the same actual quantities measured, we have

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$$\left.\begin{aligned} q\,[Q] &\equiv q'\,[Q'] \\ x\,[X] &\equiv x'\,[X'] \\ y\,[Y] &\equiv y'\,[Y'] \\ z\,[Z] &\equiv z'\,[Z'] \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots(2)$$

And since the equation between the numerical measures is by agreement the same as before, since both systems of units are absolute,

$$q' = x'^{\alpha}\, y'^{\beta}\, z'^{\gamma} \ldots;$$

$$\therefore\ \frac{q'}{q} = \left(\frac{x'}{x}\right)^{\alpha}\left(\frac{y'}{y}\right)^{\beta}\left(\frac{z'}{z}\right)^{\gamma}\ldots\ldots,$$

and hence

$$\frac{[Q']}{[Q]} = \left(\frac{[X']}{[X]}\right)^{\alpha}\left(\frac{[Y']}{[Y]}\right)^{\beta}\left(\frac{[Z']}{[Z]}\right)^{\gamma}\ldots.$$

Thus if the fundamental units X, Y, Z be changed in the ratios

$$\xi : 1, \quad \eta : 1, \quad \zeta : 1,$$

and the derived unit in the ratio $\rho : 1$, then

$$\rho = \xi^{\alpha}\, \eta^{\beta}\, \zeta^{\gamma}.$$

This statement may be evidently expressed by the relation

$$[Q] = [X]^{\alpha}\,[Y]^{\beta}\,[Z]^{\gamma} \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where, now, $[Q]$, $[X]$, $[Y]$, $[Z]$ *no longer represent concrete units but the ratios in which the derived unit* $[Q]$ *and the fundamental units* X, Y, Z *respectively are to be changed,* it being understood that the same method of defining the absolute system is to be adopted throughout.

The equation (3) giving the ratio in which a derived unit is changed when the fundamental units are changed in any given ratios is called a dimensional equation, and is very convenient for determining the factor of conversion for any unit, from one absolute system to another governed by the same principles.

The following rule for calculating the factor of conversion when the dimensional equation is given is easily remembered. If in the dimensional equation we substitute for the symbols of the fundamental units the value of each old unit in terms of the corresponding new one, the result gives the factor for converting the numerical measure of a quantity from the old system to the new.

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And if, on the other hand, for the symbols of the fundamental units there be substituted the new units in terms of the old, the result gives the factor by which the old derived unit must be multiplied to give the new derived unit.

We may under certain circumstances work backwards from the dimensional equation to the physical law, in case we know the dimensional equations from other sources. The problem in this case is practically knowing the dimensional equation for q to determine α, β, γ.

This may be applied for instance to prove the equation for the velocity of sound, viz. $v \propto \sqrt{\frac{p}{d}}$. But, as others have already remarked, we make the assumption that the velocity depends upon no other quantities but p and d, and thus get no indication of the dependence of the velocity upon either temperature or the ratio of the specific heats because neither of these quantities is measured in units which vary when the fundamental units of length, mass, and time vary.

The method of proving a physical law by means of a dimensional equation may thus be sometimes misleading. Any dimensional equation may be expressed as the product of two others, one of which may be a dimensional equation of recognizable form, but it does not necessarily follow that there is any physical interpretation corresponding to it.

The dimensional equations for electrical quantities on the electromagnetic system may be deduced from those on the electrostatic system by multiplying or dividing by a dimensional equation representing some power of a velocity.

It happens in this case that a physical meaning can be assigned to this velocity, namely, that of propagation of an electrical disturbance. But this need not always be the case; the form of the dimensional equation may be due to some hitherto unrecognized physical fact, but it may sometimes represent a relation between the ratios in which units are changed and nothing more.

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April 28, 1884.

MR GLAISHER, PRESIDENT, IN THE CHAIR.

J. C. Bose, B.A., Christ's College, was elected a fellow.

The following communications were made:

(1) *On the general equations of the electromagnetic field.* By R. T. GLAZEBROOK, M.A., F.R.S.

IN a paper read before this Society on February 25, 1884, I compared the general equations of the electromagnetic field as obtained by Maxwell and Helmholtz. The object of the present communication is to obtain the more general equations of Helmholtz in a manner analogous to that employed by Maxwell. We shall use the notation of the latter freed however from the restriction that

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \text{ is zero.}$$

We shall further assume with Maxwell that the current in a dielectric is represented by $\dot{f}$ not by $\dot{\xi}$ as in Helmholtz' work.

The electromagnetic effects depend on the values of F, G, and H the components of the electrokinetic momentum, and we have if we integrate completely round a closed curve with the usual notation

$$\int F dx + G dy + H dz = \mu \int \frac{i \cos \epsilon}{r} ds d\sigma.$$

Let ds coincide with the axes in turn, then we obtain

$$F = \mu \iiint \frac{u}{r} dx' dy' dz' + \frac{d\chi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{etc.} \ldots\ldots\ldots$$

χ being any single valued function of x, y, z and t.

Let χ be given by the equation

$$\chi = \frac{1}{4\pi} \iiint \frac{\lambda}{r} dx' dy' dz' \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

so that χ is the potential of matter of density λ distributed throughout the space. Then

$$\frac{d\chi}{dx} = \frac{1}{4\pi} \iiint \lambda \frac{x-x'}{r^3} dx' dy' dz',$$

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and transforming this by partial integration, we get

$$\frac{d\chi}{dx} = -\frac{1}{4\pi}\iiint \frac{d\lambda}{dx'} \cdot \frac{1}{r}\, dx'dy'dz' \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and as in the former paper let Φ satisfy the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \frac{1}{4\pi}\nabla^2 \frac{d\Phi}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Then we have

$$F = \mu \iiint \frac{u}{r}\, dx'dy'dz' - \frac{1}{4\pi}\iiint \frac{d\lambda}{dx'}\frac{1}{r}\, dx'dy'dz' \ldots\ldots\ldots\ldots (5),$$

etc.....

so that $$\nabla^2 F = -4\pi\mu u + \frac{d\lambda}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

etc.....

Hence if $$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\nabla^2 J = -4\pi\mu\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + \nabla^2\lambda.$$

Therefore $$\nabla^2 J + \mu\nabla^2 \frac{d\Phi}{dt} = \nabla^2\lambda \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The solution of this is

$$\lambda = J + \mu\frac{d\Phi}{dt} + V \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where $\nabla^2 V = 0$ always.

Thus V is not a function of the time and we may neglect it for our present purposes.

Hence $$\nabla^2 F - \frac{dJ}{dx} = -4\pi\mu u + \mu\frac{d^2\Phi}{dxdt} \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and substituting in the usual equations connecting α, β, γ and F, G, H

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u - \frac{d^2\Phi}{dxdt} \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

If we remember that, according to Helmholtz, $\mu k \frac{d\Phi}{dt} = -J$, these equations are identical with (10) and (15) of my former paper, which may be written

$$\nabla^2 F + \mu k \frac{d^2\Phi}{dxdt} = -4\pi\mu u + \mu\frac{d^2\Phi}{dxdt},$$

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and $$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u - \frac{d^2\Phi}{dxdt}.$$

They become identical with Maxwell's equations on putting

$$\frac{d\Phi}{dt} = 0.$$

We will consider further this case in which Φ is not zero. Let Ψ be the potential at any point arising from electrostatic forces Ω that which arises from the polarization of the dielectric. Then the equations in addition to (10) and (11) which we have to consider are

$$u = \dot{f}, \text{ etc.}$$

$$4\pi f = K\left(-\frac{d\Psi}{dx} - \frac{d\Omega}{dx} - \frac{dF}{dt}\right) \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

etc.....

Differentiate (12) with reference to x, y, z in order, and add. Then differentiating with respect to t,

$$\frac{d}{dt}(\nabla^2\Phi) = 4\pi\left(\frac{d\dot{f}}{dx} + \frac{d\dot{g}}{dy} + \frac{d\dot{h}}{dz}\right)$$

$$= -K\frac{d}{dt}\left(\nabla^2\Psi + \nabla^2\Omega + \frac{dJ}{dt}\right) \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

From (10) and (12) we find

$$\mu\frac{d^2\Phi}{dxdt} - \nabla^2 F + \frac{dJ}{dx} = -\mu K\frac{d}{dt}\left(\frac{d\Psi}{dx} + \frac{d\Omega}{dx} + \frac{dF}{dt}\right) \ldots\ldots (14),$$

etc.....

and by the usual transformation

$$\mu K\frac{d^2\alpha}{dt^2} = \nabla^2\alpha \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

and from hence by aid of the equations,

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u - \frac{d^2\Phi}{dxdt},$$

etc.....

we obtain

$$\mu K\frac{d^2}{dt^2}\left(f - \frac{1}{4\pi}\frac{d\Phi}{dx}\right) = \nabla^2\left(f - \frac{1}{4\pi}\frac{d\Phi}{dx}\right) \ldots\ldots\ldots\ldots (16).$$

etc.....

Thus the magnetic force α, β, γ, and the vector whose components are $f - \frac{1}{4\pi}\frac{d\Phi}{dx}$, etc., both travel with velocity $1/\sqrt{(\mu K)}$.

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Again substituting in (5) the value of λ which we have found in (9), we obtain as the value of F, if we omit the terms in V,

$$F = \mu \iiint \frac{u}{r}\, dx'dy'dz'$$

$$- \frac{1}{4\pi} \iiint \left(\frac{dJ}{dx'} + \mu \frac{d^2\Phi}{dx'dt} \right) \frac{1}{r}\, dx'dy'dz' \ldots\ldots\ldots\ldots\ldots (17).$$

Let us consider a plane wave of magnetic force travelling through the medium in a direction whose direction cosines are l, m, n. Let $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$ be the direction cosines of the magnetic force, then we may put

$$\alpha = A\mathfrak{L} \sin \frac{2\pi}{\lambda} (lx + my + nz - Vt) \ldots\ldots\ldots (18).$$

etc.....

Since $$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0,$$

the lines whose direction cosines are $l, m, n, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}$ respectively, are at right angles, that is, the magnetic force is in the plane of the wave.

Now let L, M, N be the direction cosines of any line normal to $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$, and ϵ be the angle between l, m, n and L, M, N, and put

$$\frac{2\pi}{\lambda} (lx + my + nz - Vt) \equiv \delta.$$

Then $$\mathfrak{L} = \frac{mN - nM}{\sin \epsilon}, \text{ etc.}$$

and $$\alpha = A \frac{(mN - nM)}{\sin \epsilon} \sin \delta = \frac{1}{\mu} \left(\frac{dH}{dy} - \frac{dG}{dz} \right) \ldots\ldots (19).$$

etc.....

These are satisfied by

$$F = - \frac{\mu A L \lambda}{2\pi \sin \epsilon} \cos \delta + \frac{d\Theta}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

etc.....

Θ being any function of x, y, z and t.

Again, since $$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u - \frac{d^2\Phi}{dxdt},$$

$$4\pi u - \frac{d^2\Phi}{dxdt} = A \frac{2\pi}{\lambda \sin \epsilon} \{m(lM - mL) - n(nL - lN)\} \cos \delta$$

$$= - \frac{2\pi A}{\lambda \sin \epsilon} \{L(l^2 + m^2 + n^2) - l(Ll + mM + nN)\} \cos \delta \ldots (21).$$

etc.....

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Now let $$u_1 = -\frac{A}{2\lambda \sin \epsilon} L \cos \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

etc.,

and $$\frac{d\Phi_1}{dt} = -\frac{A(Ll + Mm + Nn)}{\sin \epsilon} \sin \delta \quad \ldots\ldots\ldots\ldots\ldots (23).$$

These values of u_1, v_1, w_1 and Φ_1 satisfy

$$\frac{du_1}{dx} + \frac{dv_1}{dy} + \frac{dw_1}{dz} = \frac{1}{4\pi} \nabla^2 \frac{d\Phi_1}{dt},$$

we may therefore put

$$\left.\begin{aligned} u &= u_1 + u_2, \text{ etc.} \\ \Phi &= \Phi_1 + \Phi_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24),$$

where $$\frac{du_2}{dx} + \frac{dv_2}{dy} + \frac{dw_2}{dz} = \frac{1}{4\pi} \nabla^2 \frac{d\Phi_2}{dt}.$$

Substituting in (21) we have

$$u_2 = \frac{1}{4\pi} \frac{d\Phi_2}{dx\, dt}.$$

Thus the magnetic force gives rise to an electric current whose components are u_1, v_1, w_1 and direction cosines L, M, N, while there may at the same time exist an additional current u_2, v_2, w_2 independent of the magnetic force. Hence if we are given a wave of magnetic force defined by

$$\alpha = A \mathfrak{L} \sin \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

where δ is written for

$$\frac{2\pi}{\lambda} (lx + my + nz - Vt),$$

we obtain the waves defined by

$$\left.\begin{aligned} f &= \frac{A}{4\pi V \sin \epsilon} L \sin \delta + \frac{1}{4\pi} \frac{d\Phi_2}{dx} \\ F &= -\frac{\mu A \lambda}{2\pi \sin \epsilon} L \cos \delta + \frac{d\Theta}{dx} \\ \Phi &= -\frac{A\lambda}{2\pi V} \frac{(Ll + Mm + Nn)}{\sin \epsilon} \cos \delta + \Phi_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (26).$$

The quantity Φ_2 disappears from the equations which connect the displacement and the magnetic force and is independent therefore

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of the given wave of magnetic force. For the present we will omit it and put

$$\Phi = -\frac{A\lambda}{4\pi V}\frac{(Ll+Mm+Nn)}{\sin\epsilon}\cos\delta \ldots\ldots\ldots\ldots(27),$$

and then we have

$$J = -\mu\frac{d\Phi}{dt} + \nabla^2\Theta.$$

Thus from (9)

$$\nabla^2\Theta = -\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

and

$$\Theta = \frac{1}{4\pi}\iiint\frac{\lambda}{r}\,dx'dy'dz' = \chi.$$

This quantity χ disappears from the expressions for the displacement, and according to Maxwell (*Electricity and Magnetism*, II.) is not connected with any known Physical Phenomenon. He therefore neglects it. More strictly, according to Helmholtz, it may be needed to express completely the action of an element of current at a distant point, and he shews reason for putting

$$J = -\mu k\frac{d\Phi}{dt},$$

k being an indeterminate constant, and then we have

$$\nabla^2\chi = \mu(1-k)\frac{d\Phi}{dt},$$

and hence

$$\frac{d\chi}{dx} = -\mu\frac{(1-k)}{4\pi}\iiint\frac{1}{r}\frac{d^2\Phi}{dx'dt}\,dx'dy'dz'.$$

We can easily in the case before us evaluate $\frac{d\chi}{dx}$,

and we find

$$\frac{d\chi}{dx} = \frac{\mu A\lambda l}{2\pi\sin\epsilon}(1-k)(lL+mM+nN)\cos\delta,$$

and hence for the complete value of F

$$F = -\frac{\mu A\lambda}{2\pi\sin\epsilon}\{L - l(1-k)(lL+mM+nN\}\cos\delta.$$

Thus on this theory F is a vector travelling at the same rate as α, β, γ, but its direction depends on the value of k.

In the general case in which the unknown quantity χ is retained F may be divided into two parts, the one travels with the velocity V, and has for direction cosines L, M, N, that is it lies in a plane through the wave normal and at right angles to the magnetic force, the other is normal to the surface χ = constant, but its rate of motion is unknown. The direction cosines of f, g, h are also

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L, M, N, the same as that of the first part of F, G, H. There may however be additional terms in the values of f, g, h depending on Φ_2, that part of Φ which does not arise from the given wave of magnetic force.

According to Maxwell Φ is necessarily zero everywhere. Moreover (§ 616) he neglects the terms in χ, i.e. puts $J=0$ also. As was shewn in the previous paper Helmholtz' theory becomes Maxwell's by putting $\Phi=0$, while at the same time J is zero and k may have any finite value, for in that case the terms in k all disappear from the equations.

One other point may here be noticed, the values given for u, v, w and Φ agree with those found by Lorentz, equations 37, 39 of the paper referred to. According to him if

$$u=-\frac{2\pi}{T}pa\sin\frac{2\pi}{T}\left(t-\frac{lx+my+nz}{V}-p\right),$$

$$\Phi=-2TV(pl+qm+rn)\,a\sin\frac{2\pi}{T}\left(t-\frac{lx+my+nz}{V}-p\right).$$

To consider the case of a crystalline medium we have, following the methods and notation of my paper on "Some equations connected with the electromagnetic theory of light" (*Camb. Phil. Proc.* IV. Pt. iii.).

$$\frac{4\pi}{\mu}\left\{\frac{1}{K_2}\frac{dg}{dz}-\frac{1}{K_3}\frac{dh}{dy}\right\}=\frac{d\alpha}{dt} \quad \text{.....(29).}$$

Hence
$$\frac{d^2f}{dt^2}=\frac{du}{dt}=\frac{1}{4\pi}\frac{d}{dt}\left\{\frac{d^2\Phi}{dxdt}+\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right\},$$

$$\frac{d^2}{dt^2}\left(f-\frac{1}{4\pi}\frac{d\Phi}{dx}\right)$$
$$=\bar{a}^2\left(\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}\right)-\bar{b}^2\frac{d^2g}{dxdy}-\bar{c}^2\frac{d^2h}{dzdx} \quad \text{.....(30),}$$

etc.....

These give us the equations satisfied by f, g, h.

Also

$$\frac{d^2\alpha}{dt^2}=\frac{4\pi}{\mu}\left\{\frac{1}{K_2}\frac{dv}{dz}-\frac{1}{K_3}\frac{dw}{dy}\right\}$$
$$=\frac{1}{\mu K_2}\frac{d}{dz}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)-\frac{1}{\mu K_3}\frac{d}{dy}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)$$
$$+\frac{d^2\Phi}{dtdydz}\left\{\frac{1}{\mu K_2}-\frac{1}{\mu K_3}\right\}=\bar{b}^2\frac{d^2\alpha}{dz^2}+\bar{c}^2\frac{d^2\alpha}{dy^2}$$
$$-\bar{b}^2\frac{d^2\gamma}{dzdx}-\bar{c}^2\frac{d^2\beta}{dxdy}+(\bar{b}^2-\bar{c}^2)\frac{d^2\Phi}{dtdydz} \quad \text{.....(31).}$$

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We have seen that in an isotropic medium a wave of electric displacement is propagated with a velocity $1/\sqrt{\mu K}$. Let us suppose that in the crystal there can also be a wave of electric displacement given by

$$f = BL \sin \delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

etc.

where $$\delta = \frac{2\pi}{\lambda}(ln + my + nz - Vt).$$

Then $$\Phi = -2\lambda B \cos \epsilon \cos \delta \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

and we require to find the relations between the quantities

$$V,\ L,\ M,\ N,\ l,\ m,\ n.$$

Substituting in the equations for f, g, h we have

$$L(m^2+n^2)(V^2-\bar{a}^2) - Mlm(V^2-\bar{b}^2) - Nln(V^2-\bar{c}^2) = 0 \ldots\ldots\ldots (34),$$

and two similar equations. On multiplying by l, m, n and adding the sum is zero, there are therefore only two independent equations and these are insufficient to determine V, L, M and N. Thus the problem as it stands is indeterminate, another equation is required for its solution.

Let $$\bar{a}^2 Ll + \bar{b}^2 Mm + \bar{c}^2 Nn = q^2 \cos \zeta,$$

q^2 being a quantity to be further defined shortly. Then we find

$$V^2 = \frac{L\bar{a}^2(m^2+n^2) - M\bar{b}^2 lm - N\bar{c}^2 nl}{L(m^2+n^2) - Mlm - Nnl} = \frac{L\bar{a}^2 - q^2 \cos \zeta}{L - l \cos \epsilon} \ldots\ldots (35).$$

The other two equations can be treated similarly; and we find finally

$$\frac{L\bar{a}^2 - lq^2 \cos \zeta}{L - l \cos \delta} = \frac{M\bar{b}^2 - mq^2 \cos \zeta}{M - m \cos \delta} = \frac{N\bar{c}^2 - nq^2 \cos \zeta}{N - n \cos \delta} \ldots\ldots\ldots (36).$$

On eliminating $q^2 \cos \zeta$ from the first and second and first and third respectively, we get

$$(L - l \cos \delta)\{lMN(\bar{b}^2 - \bar{c}^2) + mNL(\bar{c}^2 - \bar{a}^2) + nLM(\bar{a}^2 - \bar{b}^2)\} = 0 \ldots\ldots\ldots\ldots (37).$$

Hence we find

$$\cos \delta = \frac{l}{L} = \frac{m}{M} = \frac{n}{N} = 1,$$

or $$\frac{l}{L}(\bar{b}^2 - \bar{c}^2) + \frac{m}{M}(\bar{c}^2 - \bar{a}^2) + \frac{n}{N}(\bar{a}^2 - \bar{b}^2) = 0 \ldots\ldots\ldots (38).$$

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Now the equations $L=l$, $M=m$, $N=n$, when substituted in (34) lead to

$$(m^2+n^2)(V^2-\bar{a}^2)-m^2(V^2-\bar{b}^2)-n^2(V^2-\bar{c}^2)=0,$$

or
$$\bar{a}^2(m^2+n^2)=\bar{b}^2m^2+\bar{c}^2n^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

and two similar equations. This is impossible, thus we must have

$$\frac{l}{L}(\bar{b}^2-\bar{c}^2)+\frac{m}{M}(\bar{c}^2-\bar{a}^2)+\frac{n}{N}(\bar{a}^2-\bar{b}^2)=0,$$

and this is the only relation between l, m, n and L, M, N.

The fundamental equations may as we have seen, be written

$$f=-\frac{K_1}{4\pi}\left(\frac{d\Psi}{dx}+\frac{d\Omega}{dx}+\frac{dF}{dt}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (40).$$

etc.....

Hence

$$4\pi\left\{\frac{1}{K_1}\frac{df}{dx}+\frac{1}{K_2}\frac{dg}{dy}+\frac{1}{K_3}\frac{dh}{dz}\right\}+\left(\nabla^2\Psi+\nabla^2\Omega+\frac{dJ}{dt}\right)=0 \ldots (41).$$

Lorentz in the paper referred to, following Helmholtz in measuring the current by $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, puts $\Omega=\Phi$, and $J=-\mu k\dfrac{d\Phi}{dt}$.

$\nabla^2\Psi$ is zero for the electrostatic forces arise by hypothesis from electricity outside the space considered, and the equation thus modified becomes his fourth equation. We have in the former paper seen reasons against putting $\Omega=\Phi$ when we adopt Maxwell's definition of the current.

Again, from the assumed values for f, g, h, we have, as in the paper in the *Camb. Phil. Proc.* IV.

$$\frac{d\alpha}{dt}=4\pi\left\{\bar{b}^2\frac{dg}{dz}-\bar{c}^2\frac{dh}{dy}\right\},$$

$$\frac{d\alpha}{dt}=4\pi\frac{2\pi}{\lambda}B\{\bar{b}^2Mn-\bar{c}^2Nm\}\cos\delta,$$

and
$$\alpha=-\frac{4\pi B}{V}\{\bar{b}^2Mn-\bar{c}^2Nm\}\sin\delta \quad \ldots\ldots\ldots\ldots\ldots (42),$$

also
$$l\alpha+m\beta+n\gamma=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43),$$

thus α, β, γ is in the wave front.

Again $L\alpha+M\beta+N\gamma=0$ in virtue of (38). Hence α, β, γ is at right angles to L, M, N. Also multiplying by $\bar{a}^2L$, $\bar{b}^2M$ and $\bar{c}^2N$ and adding the sum is zero. Hence α, β, γ is at right angles

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to the line whose direction cosines are proportional to $\bar{a}^2L$, $\bar{b}^2M$ and $\bar{c}^2N$.

From this it follows easily that if we construct the ellipsoid $\bar{a}^2x^2+\bar{b}^2y^2+\bar{c}^2z^2=1$, and take any radius vector length r as the direction of L, M, N then this direction and the direction of the magnetic force are axes of the section of the ellipsoid which is formed by the plane passing through them; while *any line in the plane through* L, M, N *at right angles to this plane section* may be the wave normal. Thus so far as the equations hitherto considered are concerned, there may be for any given possible direction of electric displacement L, M, N an infinite number of possible wave normals all lying in the plane

$$\frac{x}{L}(\bar{b}^2-\bar{c}^2)+\frac{y}{M}(\bar{c}^2-\bar{a}^2)+\frac{z}{N}(\bar{a}^2-\bar{b}^2)=0,$$

or for any given wave normal an infinite number of possible directions of electric displacement lying on the cone

$$\frac{l}{x}(\bar{b}^2-\bar{c}^2)+\frac{m}{y}(\bar{c}^2-\bar{a}^2)+\frac{n}{z}(\bar{a}^2-\bar{b}^2)=0.$$

The value of V is given in equations (35), a simpler form is obtained as follows. We have seen above, that if $f=BL\sin\delta$, etc.,

then $$\alpha=4\pi VB(mN-nM)\sin\delta \quad \ldots\ldots(44),$$

etc.

$$\Phi=-2\lambda B\cos\epsilon\cos\delta.$$

Comparing this value of α with that given in (42), we have

$$V^2=\frac{\bar{b}^2Mn-\bar{c}^2Nm}{Mn-Nm}=\text{etc.} \quad \ldots\ldots(45).$$

Now, if p is the length of the perpendicular on the tangent plane, at the extremity of a radius vector of length r in direction L, M, N, then the direction cosines of p are $pr\bar{a}^2L$, etc., and $q^2=1/pr$.

The direction of the magnetic force is, we have seen, at right angles to this perpendicular, to the line l, m, n, and to the line L, M, N; ϵ is the angle between l, m, n and L, M, N, ζ that between l, m, n and the perpendicular. Thus we have, expressing the direction cosines of α, β, γ in the three possible ways,

$$\frac{Mn-Nm}{\sin\epsilon}=\frac{pr(\bar{b}^2Mn-\bar{c}^2Nm)}{\sin\zeta}=\frac{prMN(\bar{b}^2-\bar{c}^2)}{\sin(\epsilon-\zeta)} \quad \ldots(46).$$

Also $$p=r\cos(\epsilon-\zeta) \quad \ldots\ldots(47).$$

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Hence, from (45) we obtain

$$V^2 = \frac{\sin\zeta}{pr\sin\epsilon} = \frac{1}{r^2}\frac{\sin\zeta}{\sin\epsilon\cos(\epsilon-\zeta)} \quad \text{.....(48).}$$

We have also a third expression for α, viz.

$$\alpha = -4\pi V.B.MN(\bar{b}^2-\bar{c}^2)r^2\sin\epsilon\cot(\epsilon-\zeta)$$

$$= -\frac{4\pi BMN(\bar{b}^2-\bar{c}^2)\sin\zeta}{V\sin(\epsilon-\zeta)} \quad \text{.....(49).}$$

This is as far as we can carry the theory with the equations at present considered. But α, β, γ have to satisfy equations (31). We must therefore investigate the conditions which this gives rise to.

Now it is clear that the value of α already found, will satisfy (31), for the value in (42) is obtained from

$$\frac{d^2\alpha}{dt^2} = 4\pi\left(\bar{b}^2\frac{dv}{dz} - \bar{c}^2\frac{dw}{dy}\right), \text{ etc.}$$

while the values in (44) come from

$$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx} + \frac{d^2\Phi}{dtdy}, \text{ etc.,}$$

and these are the equations used in forming (31).

Thus the values found satisfy (31) without any fresh conditions. So that without some other condition the problem of the propagation of an electromagnetic disturbance in a crystal is indeterminate.

For a given direction of displacement we have a definite velocity and a definite magnetic force, but an infinite number of plane wave fronts, all of which pass through the direction of the magnetic force; for a given direction of wave propagation we have an infinite number of directions of electric displacement lying on a certain cone, and to each of these there corresponds a definite velocity and a definite direction for the magnetic force.

Now experiments shew that the velocity of a plane wave of light in a crystal has one of two definite values, and that these values agree very closely with those given by Fresnel's theory.

Let us assume that these conditions hold for the electromagnetic displacement; then since we are to have two values each for L, M, N, when l, m, n are given L, M, N lies on a cone of the second degree with its vertex at the origin the additional relation between L, M and N must be of the form

$$PL + QM + RN = 0,$$

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where P, Q and R are any constants; Fresnel's construction will be satisfied if

$$Ll + Mm + Nn = 0,$$

that is if the displacement is in the wave front. This condition is equivalent to

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0,$$

which of course is Maxwell's fundamental assumption.

Thus in order to deduce Fresnel's construction from the electromagnetic theory of light the condition

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0$$

is *necessary* and sufficient. Maxwell himself had of course shewn the sufficiency of the condition, but I am not aware that the problem has been treated previously quite so generally as in the present paper.

(2) *On the pulsations of Spheres in an Elastic Medium.* *By* A. H. Leahy, B.A.

[*Abstract.*]

The problem of two pulsating spheres in an incompressible fluid has been discussed by several writers. The author considers the analogous problem in the case in which the medium surrounding the spheres has the properties of an elastic solid. He finds that the most important term in the expression of the law of force between the two spheres varies inversely as the square of the distance between them. This force will be an attraction if the spheres be in unlike phases, a repulsion if they be in like phases at any instant. The next term in the expression varies inversely as the cube of the distance between the two spheres and is always a repulsion.

The paper is being published in full in the *Transactions* of the Society.

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May 12, 1884.

MR GLAISHER, PRESIDENT, IN THE CHAIR.

The following were elected honorary members of the Society:

A. Baeyer, Professor of Chemistry at Munich.

Anton Dohrn, Director of the Zoological Station at Naples.

Carl Gegenbaur, Professor of Comparative Anatomy in the University of Heidelberg.

G. Mittag-Leffler, Professor of Mathematics in Stockholm.

E. F. W. Pflüger, Professor of Physiology in the University of Bonn.

Gustav Quincke, Professor of Physics in the University of Heidelberg.

H. A. Rowland, Professor of Physics in the Johns-Hopkins University, Baltimore, U.S.A.

Julius Sachs, Professor of Botany in the University of Würtzburg.

H. G. Zeuthen, Professor of Mathematics in Copenhagen.

R. Stawell Ball, Astronomer Royal for Ireland.

W. T. Thiselton Dyer, Assistant Director of the Royal Gardens, Kew.

J. Whitaker Hulke, Ex-President of the Geological Society.

The following communication was made:

(1) *On a Continuous Succession in part of the Guernsey Gneiss. By* Rev. E. HILL, M.A.

IN the discussion which followed a paper 'on the Rocks of Guernsey' read by me before the Geological Society it was suggested by one of the speakers that appearances in Archaean rocks of conformable succession and enormous thickness are often deceptive, and result from repeated plications of a moderately thick series. I have no doubt that this remark is correct, and it is perfectly possible that such plications may have happened in Guernsey, and may hereafter by careful examination be detected. But notwithstanding there cannot be any doubt that the Guernsey rocks do disclose a series of very great thickness.

While making a specially careful examination of a locality in Guernsey in search of evidence for or against the faulted position of certain unusual rocks, I began to notice that particular varieties of the gneiss could be recognized at various localities as agreeing together. In some of the outcrops a distinct strike of

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the foliation was visible, and the positions of the outcrops which resembled each other harmonized with this strike. Pursuing the investigation I found that there was evidence of a complete and conformable succession over a considerable distance. I think that it may be interesting to briefly note this, and to remark upon one or two conclusions which follow from it.

The nature of the succession and the evidence for it need not be given very fully. All the beds are gneiss of the very oldest type. The strike of the foliation can usually be detected, and is always in the region dealt with N. and S. The dip is sometimes nearly vertical, but generally more or less steeply inclined, invariably to the W. Five successive distinct series can be clearly recognized, the first along Vazon Bay and at several points to the S., the second along Perelle Bay, the third around the L'Erée Hotel, the fourth on the L'Erée peninsula and across the sound of Lihou, the fifth in the island of Lihou. I have not seen any passage of the first into the second, but the second can be traced continuously in the shore at low water, and seen to change into the third along the ridge extending out into the sea to the islet called Chapelle Domhue. The third can be seen passing into the fourth on the L'Erée shore, and also in the outermost rocks of the islet just mentioned. The fourth is seen within 10 feet of the fifth on the shore of Lihou, separated only by one of the greenstone dykes which are so numerous.

Now were the beds of this series of rocks repeated either by faults or by plications and over-fold, then as we made traverses in directions across their strike we should from time to time often find repetitions of particular beds. But in the succession above described nothing of the kind is seen. Each division of the series is perfectly distinct from the rest, lasts for a certain space, is succeeded by another, and does not, in the area described at least, reappear. It would be possible that any one of these divisions separately might owe its thickness to plication, but I cannot conceive this process going on in more than one without an obvious repetition of alternating divisions. We may conclude that the thickness of the series as estimated by this area is a real thickness free from the errors introduced by plications. There may perhaps be an error due to irregular thickness or lenticular arrangement, but I am not now considering these. Moreover I have identified series II. and III. for more than two miles along their strikes, and I. at extreme points separated by three miles.

The breadth of this succession measured across the strikes is over two and a half miles. Allowing for the inclination of the beds this must represent a thickness of nearly two miles.

Another conclusion I should draw is that the nature of gneiss must depend not so much on metamorphosing agents as on the

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materials constituting the original beds. For II. and IV. are in parts so highly crystalline that they begin to resemble an igneous rock, while III. has no such resemblance, nor yet I.; while V. is very markedly sedimentary. Thus we have two extremely crystalline series intercalated between others far less so. The difference in character cannot be due therefore to any such cause as one series having been subjected to a greater central heat than another, or to contact metamorphism, or to heat developed in compression. The dykes, which are numerous, penetrate all alike. The neighbouring granite is equally remote from all. The compression must have operated equally on all. Thus the only explanation for the distinctness of these several series seems original difference of constitution.

In a paper recently published by Mr Marr he suggests the subaërial and volcanic origin of Archaean rocks. The facts which I have noticed agree pretty well with subaërial origin. The general arrangement of the constituents rather into lenticular masses than laminae, the occasional larger lenticular masses or short seams, and the frequent appearances of false-bedding, where the bedding is sufficiently distinct, all point in that direction. So perhaps may the frequent and well-marked nodes or patches, which abound in the highly crystalline Series II. and exactly resemble those so common in granites.

But in favour of the volcanic origin I have no facts to adduce. I have seen no indications of agglomeratic constitution nor of included fragments. And I think some such might be expected. I am more inclined to go back to the older hypothesis, and imagine that the Archaean rocks took their origin when the surface of the earth was in some respects exposed to different conditions than those which we now experience.

May 26, 1884.

MR GLAISHER, PRESIDENT, IN THE CHAIR.

Prof. E. Ray Lankester was elected an honorary member.
S. L. Hart, M.A., St John's College, was elected a Fellow.

The following communications were made:

(1) *On some Irregularities in the Values of the Mean Density of the Earth, as determined by Baily.* *By* W. M. HICKS.

I HAVE recently been examining Baily's observations on the mean density of the earth in order to see if they showed any traces of a dependence of the attraction between two masses on their

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temperature. I was astonished to find in his numbers most decided signs of some temperature effect, and although I have not been able to discover the cause of this effect, yet it may be important to put the facts on record. The description of the apparatus, calculations, and observations form Vol. XIV. of the *Memoirs of the Royal Astronomical Society*, which also contains the mathematical theory as developed by Mr (now Sir George) Airy. His observations extended over sixteen months from January 24, 1841, to May 8, 1842, and amounted in all to about 2000 separate sets of observations. They were therefore made at all temperatures, varying from a winter temperature of 30° F. to a summer temperature of 69° F. They were made by employing balls of various materials for the torsion balance—lead, platinum, zinc, brass, hollow brass, glass and ivory—of different sizes, and suspended in different ways,—bifilar lines of silk and metal at different distances, and single wires of different metals and diameters. These were all arranged in sixty-two different sets. To discover any temperature effects, it would have been better had the experiments been carried out on one uniform system; but as it is, those averages will give the most reliability for comparison, which are based on the largest number of different series, as thereby we eliminate effects due to particular series.

The method adopted in forming the table below has been first to go through Baily's Table I. and make lists of those days when the temperature lay within certain limits, thus for instance two degrees above or below 50°. On any particular day the temperature in the case, which surrounded the box in which the torsion rod was suspended, remained remarkably constant, only as a rule varying a fraction of a degree. Within the torsion box it would therefore be almost rigorously uniform. By means of the preliminary lists, the deduced daily means for those chosen days were then extracted from his Table II. and entered in lists by themselves. Then the means of the daily means in each series were taken, and finally the means of these last taken. In this way, irregularities due to special days and to different series were successively smoothed down as far as possible. The results are given in the table below. I have not discussed the whole of the observations, but only those near the temperatures marked. For the lowest temperatures of all, from 30° to 37° I have not taken account of his second series of experiments. My reason for this is that the method by which the observations were made was different from all the others, and a bad one. Also that out of 12 means for these temperatures, 5 would be from this series, which happened to be made in cold weather. The consequence of admitting them would therefore be to mask effects due to temperature by the errors of observation of this particular series. The densities obtained by it

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vary from 5·2886 to 5·8717, Baily himself says they are "not reliable," and says,—"I made only 23 experiments in this way, for although the principle may be correct, yet the difficulty of seizing the precise moment at which the masses ought to be put in motion occasionally introduced disturbances which affected the accuracy of the results;" and later, "it may become a question whether these experiments are entitled to that confidence which the subject requires." If we had a large number of series from which to take our means it might be well to retain them, but naturally at so low a temperature only a few observations were to be expected. The results of the discussion so far as it has been carried, are given in the following table. The first column gives the temperature, the second the number of series on which the mean given is based, the third the number of daily means, and the fourth the number of separate means employed. The means for the different temperatures are given in the fifth column.

Temperature F.	Number of series.	Number of daily means.	Number of observations.	Mean density.
				5 +
36° (mean)	4	7	46	·7296
40 ± 2	12	22	128	·7341
45 ± 2	20	43	247	·6823
50 ± 2	18	38	302	·6799
55 ± 2	12	23	187	·6594
60 ± 2	13	31	333	·6495
65 ± 2	7	17	140	·5935
68 (mean)	4	11	96	·5828

The gradual fall of mean density with rise of temperature is most marked, the only exception being in the case of the lowest temperature (36°) which is slightly smaller than for the temperature of 40°. However, this is based on only four series, but is nevertheless higher than for any temperatures above 40°. It is easily seen from the irregularity at which the means decrease that all disturbances have not been eliminated; still the numbers suffice to shew that some cause depending on temperature has been at work to produce disturbances. Although I undertook the discussion for the very purpose of seeing whether such an effect was present, I must confess I was astonished to find it so marked. It certainly seems too large to be explained by supposing the gravitation between two bodies to be a function of their temperatures, although it is in the direction which we should expect to find, if the gravitation increased with the temperature. For if so, we should suppose from our experiments at a high

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temperature that matter attracted itself more strongly than if we made experiments at a low temperature; consequently the mass of the earth would not require to be so large in the first case as in the second, to produce the same force on a gramme at its surface, or its mean density would be less.

In searching for some possible explanation of this temperature effect, I have been led to notice a point in the theory of the experiments which the then Astronomer Royal, Sir G. Airy, had not taken notice of. The correction for this is more important than some others which have been taken account of, introducing an alteration in the third place of decimals, while the calculations are carried to the fourth. In calculating the attraction of the masses, he has neglected to consider that of the air displaced by the masses attracting. The effect of this is to decrease the apparent attracting masses by the masses of the air displaced. The correction to be applied is easily determined. Let ρ be the density of the air, σ of the large masses (lead in all the experiments), σ' of the balls on the torsion rod, Δ the mean density of the earth as given in the tables, and Δ_1 the corrected value of Δ. Then for this term we may neglect roughly the effect of the support of the lead masses, and the torsion rods. In this case

$$\Delta_1 = \left(1 - \frac{\rho}{\sigma}\right)\left(1 - \frac{\rho}{\sigma'}\right)\Delta = \left\{1 - \rho\left(\frac{1}{\sigma} + \frac{1}{\sigma'}\right)\right\}\Delta,$$

at zero temperature, $\rho = \cdot00129$, $\sigma = 11\cdot4$, whence

$$\Delta_1 - \Delta = -\left(\cdot000113 + \frac{\cdot00129}{\sigma'}\right) \times 5\cdot67,$$

$$= -\left(\cdot00064 + \frac{\cdot0073}{\sigma'}\right).$$

From this, the corrections to be applied in the case of the different balls are found to be

Pt, $-\cdot0010$, Pb, $-\cdot0013$, Zn, $-\cdot0017$, Brass, $-\cdot0015$, Glass, $-\cdot0031$, Ivory, $-\cdot0045$.

In addition to this another correction ought to be applied, due to the inertia of the air in which the balls move. For the balls alone this would be determined, if there were no boundaries, by supposing their masses increased by half that of the air displaced. The effect of this would also be to make the true densities less than those given by $\cdot0036/\sigma'$, on the supposition that the effect on the rod itself would be much smaller in comparison. The effect of the friction of the air would be eliminated by the method of observing the time of vibration. These corrections tend to explain

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why the mean density calculated from ivory and glass balls is higher than the mean.

All these effects would vary with the temperature, but the corrections due to those variations although in the proper direction are so small that they produce no effect to the fourth place of decimals. It is difficult to conceive that the temperature-effects shown by the table are due to convection currents in the torsion box, for the closed box is itself within a case, within which the temperature was found to vary extremely slightly during a day's experiments. It is natural to look to changes in the geometrical dispositions of the parts due to changes of temperature, as a source of the effects. The expansion of the wood work would be so small as scarcely to produce any perceptible change. The only metallic expansions that occur to me as likely to produce alterations, are (α) in the length of the suspension wires, (β) the brass scale from which the deviations were read, (γ) the brass rod on which the microscopes were mounted to measure the distance of the centres of the masses and balls, (δ) and the error introduced in measuring this distance due to taking the differences between the surfaces and allowing for the radii, which would vary with the temperature. It may be well to notice each of these in order.

(α) The centres of the attracting masses are on a level with the centres of the balls; consequently if these last be raised or lowered slightly by a contraction or extension of the suspension, the effect would be vanishingly small. The alteration in the attraction of the supports would also be a negligeable part of the whole.

(β) With a rise of temperature the scale expands; consequently the reading for the deviation of the torsion rod in the two positions of the masses will be too small. We therefore suppose that the gravitational unit of force is smaller than it is, and thus make the mean density of the earth larger. For this case therefore the deduced mean densities would increase with the temperature—in the opposite direction to what is actually the case. The correction to be applied for a temperature $t°$ F. above the mean would be $-5{\cdot}67 \times {\cdot}0000104t = -{\cdot}000059\,t$.

(γ) The distances between the masses and balls were measured by microscopes fixed to a brass bar, at a distance of 5 inches from one another, which was roughly about the distance of the surface of the attracting mass from the centre of a ball. The difference from 5 in. of the actual distance was then measured by means of a small pearl scale in the microscope. Hence a correction must be introduced for the expansion of the brass bar. The reading will make the distance too small by $5\alpha t$ where α is the coefficient of linear expansion for brass, and t is the difference between the temperature of observation, and the temperature at which the distance of the microscopes is exactly 5 inches.

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(δ) The Expansion of the lead mass will produce a similar effect. Its radius was 6 in., consequently the reading will be too small by $6\beta t_1$, where β is the coefficient of expansion for lead, and t_1 is the excess of temperature over that at which the radius is 6 inches. Now allowance is made for the variation of distance from 11 inches. If the temperatures are measured from the temperature at which the radii were measured, the whole apparent diminution of distance due to causes (γ) and (δ) will be $(5\alpha + 6\beta)t$, that is, the distance instead of being δ as in Baily's formula, will be

$$\delta - (5\alpha + 6\beta)\, t.$$

Baily calculates the mean density for a distance of 11 inches, and corrects by multiplying the result by $(\delta/11)^2$. The correction to this multiplier will be therefore $-2\delta\,(5\alpha + 6\beta)\,t/11^2$, or since $\delta = 11$ very nearly $= -2\,(5\alpha + 6\beta)\,t/11$. Substituting numbers and taking the mean density to be 5·67, the correction is

$$\Delta_1 - \Delta = -\cdot 000063\, t.$$

The difference between two of the mean densities obtained, without taking account of corrections β, γ, δ, at temperatures whose difference is t, is therefore $-\cdot 000122\, t$, that at the higher temperature being the greater. These corrections would make the table above show still more striking results; e.g. increasing the difference at 68° over that at 36° by about ·0036. No other possible cause to produce temperature effects occurs to me. Is it possible that Baily's personal equation was a function of the temperature? If his judgment became more bountiful as the air became warmer, the error in the mean densities would be in the direction indicated by the table given above.

(2) *On Possible Systems of Jointed Wickerwork, and their Degrees of Internal Freedom.* *By* J. LARMOR, M.A.

IF the two sets of generating lines of a hyperboloid of one sheet be constructed by rods jointed where they cross one another, the system so formed will not be stiff. This statement is verified by the simplest examination of an ordinary paper-basket, or—much better—of one of the jointed frameworks of wooden rods that are sometimes placed round flower-pots.

Mr A. G. Greenhill has remarked (Math. Tripos, 1879) that the forms assumed by the framework on deformation are those of a confocal system of hyperboloids. This result may be proved synthetically as follows. Consider such a confocal system in position; to the points which lie on a straight line on one of them there correspond (in Ivory's manner) points on any other, which also lie

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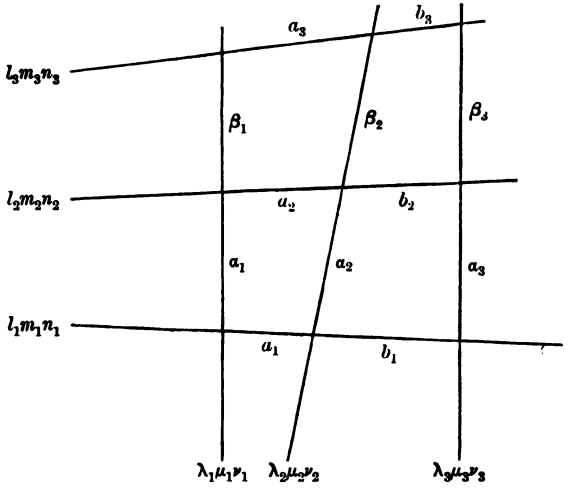
on a straight line, since the correspondence is of the first order or linear. Thus to a generator corresponds a generator, and the points of intersection of pairs of generators also correspond. Again, all points corresponding to a given point lie on a curve which cuts the system of surfaces normally, being in fact the curve of intersection of two confocals of the other kinds. So that if we consider any generator and the corresponding one on a consecutive surface, the lines joining their extremities (where they meet generators of the other system) are normal to the surface, and therefore to the generators, and the generators are therefore of equal length[1]. The condition necessary for deformation is thus satisfied, and the surface, supposed made up of jointed rods, may be deformed without straining into the consecutive confocal surface, and therefore by successive steps, into any other confocal surface.

We propose to investigate directly the cause of this want of stiffness, and to determine the number of degrees of internal freedom possessed by other systems (which we shall prove to exist) composed of three sets of rods connected by ball-joints, there being three rods at each joint.

In discussing the first problem, we may confine our attention to three rods crossing three other rods: for we shall prove that



every other rod that crosses one set of them meets each rod that crosses the other set, at a point *in the rod* which is unaltered by the deformation. And for similar reasons, we shall only have to consider in the second case the quasi-cubical framework formed

[1] Cf. H. J. S. Smith, *Proc. Lond. Math. Soc.* II. p. 244.

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by three such sets of six placed one above the other, and tied together by nine rods passing through them, making twenty-seven rods in all.

Consider then a system of six, represented on the flat by the annexed scheme, in which their direction cosines to fixed axes $l_1 m_1 n_1, \ldots$, $\lambda_1 \mu_1 \nu_1, \ldots$ are indicated, and also the lengths of the three segments of each a_1, b_1, c_1 $(= a_1 + b_1)$, ... α_1, β_1, γ_1 $(= \alpha_1 + \beta_1)$,... By expressing that the projections of each independent circuit on the axes of co-ordinates are zero, we exhaust *all* the independent relations of the system. We thus obtain four sets of three equations each, of which the following is one:

$$\left.\begin{aligned} a_1 l_1 - a_2 l_2 &= \alpha_1 \lambda_1 - \alpha_2 \lambda_2 \\ a_1 m_1 - a_2 m_2 &= \alpha_1 \mu_1 - \alpha_2 \mu_2 \\ a_1 n_1 - a_2 n_2 &= \alpha_1 \nu_1 - \alpha_2 \nu_2 \end{aligned}\right\},$$

and we have in addition six equations of the type

$$l^2 + m^2 + n^2 = 1.$$

Thus we have 18 equations between 18 variables. But these variables involve 3 indeterminates, depending on the directions of the axes: and as we know that the system is not rigid, there is a fourth indeterminate. Therefore the 18 equations are equivalent to only 14 independent equations, and that can only be by reason of the existence of 4 relations between the coefficients, i.e. between the lengths of the segments of the rods. And, conversely, if we obtain these four relations independently, we can infer that the jointed system is not rigid.

We can readily obtain them as follows:—Let $\overset{\wedge}{a_1 \alpha_1}$ denote the angle between the lines a_1, α_1: then by equating two expressions for the square of the diagonal of the reticulation $a_1 \alpha_1 a_2 \alpha_2$ we obtain

$$a_1^2 + \alpha_1^2 - 2 a_1 \alpha_1 \cos \overset{\wedge}{a_1 \alpha_1} = a_2^2 + \alpha_2^2 - 2 a_2 \alpha_2 \cos \overset{\wedge}{a_2 \alpha_2},$$

or
$$a_1 \alpha_1 \cos \overset{\wedge}{a_1 \alpha_1} - a_2 \alpha_2 \cos \overset{\wedge}{a_2 \alpha_2} = \tfrac{1}{2}(a_1^2 + \alpha_1^2 - a_2^2 - \alpha_2^2),$$

and similarly

$$b_3 \beta_3 \cos \overset{\wedge}{b_3 \beta_3} - b_2 \beta_2 \cos \overset{\wedge}{b_2 \beta_2} = \tfrac{1}{2}(b_3^2 + \beta_3^2 - b_2^2 - \beta_2^2),$$

$$c_1 \gamma_1 \cos \overset{\wedge}{c_1 \gamma_1} - c_3 \gamma_3 \cos \overset{\wedge}{c_3 \gamma_3} = \tfrac{1}{2}(c_1^2 + \gamma_1^2 - c_3^2 - \gamma_3^2),$$

three equations between the cosines of the angles $\overset{\wedge}{a_1 \alpha_1}$, $\overset{\wedge}{b_3 \beta_3}$, $\overset{\wedge}{c_3 \gamma_3}$. But we know that these angles are not determinate, therefore the result of eliminating $\cos \overset{\wedge}{b_3 \beta_3}$ between the two first equations must be equivalent to the third. That result is

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$$a_1\alpha_1 . b_2\beta_2 \cos \overset{\wedge}{c_1\gamma_1} - a_2\alpha_2 . b_3\beta_3 \cos \overset{\wedge}{c_3\gamma_3}$$
$$= \tfrac{1}{2} a_2\alpha_2 (b_2^2 + \beta_2^2 - b_3^2 - \beta_3^2) - \tfrac{1}{2} b_2\beta_2 (a_2^2 + \alpha_2^2 - a_1^2 - \alpha_1^2),$$

and we therefore have

$$\frac{a_1\alpha_1 . b_2\beta_2}{c_1\gamma_1} = \frac{a_2\alpha_2 . b_3\beta_3}{c_3\gamma_3} = \frac{\cdots\cdots}{\cdots\cdots},$$

therefore $$a_1 b_2 c_3 . \alpha_1\beta_2\gamma_3 = a_3 b_2 c_1 . \alpha_3\beta_2\gamma_1.$$

And in the same way we can obtain three other similar relations, thus making up the four relations required.

Having now obtained these relations between the segments of two triads of mutually intersecting lines in space, we may easily verify their truth in other ways. We notice that they are projective for the same reason that anharmonic ratios are projective. Projecting therefore on the principal plane of the hyperboloid to which they belong, we have two triads of tangent lines to a plane ellipse. We can now project the ellipse into a parabola. But three fixed tangents to a parabola cut all variable tangents similarly, since they with the tangent line at infinity cut them in a constant anharmonic ratio: hence now

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2 = a_3 : b_3 : c_3,$$
$$\alpha_1 : \beta_1 : \gamma_1 = \alpha_2 : \beta_2 : \gamma_2 = \alpha_3 : \beta_3 : \gamma_3,$$

and the relations are obviously true.

[We may express this argument differently by changing the hyperboloid into a hyperbolic paraboloid by a linear transformation (which we may call a projection in space of four dimensions), and noticing that the theorems are true for the paraboloid because the generators of one system divide all those of the other system similarly.]

It is to be noticed that they are not true in general for two triads in a plane: also, inasmuch as there is only one condition necessary that six lines should touch a conic, that three *other* relations do hold in a plane.

When three lines cross three other lines in a plane the three relations between the segments formed are however still true for lines crossing in space, and are moreover clearly of a projective character. We may obtain one of them as follows. From the equations between the cosines already given, we find

$$\cos \overset{\wedge}{a_2\alpha_2} = \frac{\Delta'}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_1\alpha_1 & -a_2\alpha_2 & 0 \\ 0 & -b_2\beta_2 & b_3\beta_3 \\ c_1\gamma_1 & 0 & -c_3\gamma_3 \end{vmatrix}$$

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$$= a_1b_2c_3 \cdot \alpha_1\beta_2\gamma_3 - a_2b_3c_1 \cdot \alpha_2\beta_3\gamma_1,$$

$$\Delta' = b_3c_1\beta_3\gamma_1 (a_1^2 + \alpha_1^2 - a_3^2 - \alpha_3^2)$$
$$+ c_3a_1\gamma_3\alpha_1 (b_2^2 + \beta_2^2 - b_3^2 - \beta_3^2)$$
$$+ a_1b_3\alpha_1\beta_3(c_3^2 + \gamma_3^2 - c_1^2 - \gamma_1^2).$$

Proceeding in a similar manner we find a like expression for $\cos \overset{\wedge}{b_2\alpha_2}$. Therefore, since $\cos \overset{\wedge}{b_2\alpha_2} = -\cos \overset{\wedge}{a_2\alpha_2}$ we obtain a relation of the twelfth degree between the 12 segments: and we may find two other similar ones in the same way. It is to be noticed that the diagram is not really symmetrical, so that we cannot proceed from one expression to another by simple permutation of the symbols.

Having thus independently established the existence of these four relations, we establish at the same time the flexibility of the system of six rods. Now every line that crosses three of the rods meets every line that crosses the other three. For, if we denote the two systems of rods for an instant by 123... 1′2′3′... the planes through 1′ and 1234 cut all lines in a constant anharmonic ratio, therefore 123 are each divided in the same anharmonic ratio by 1′2′3′4′. Now consider 4, which is drawn across 1′2′3′: the plane 14′ with the lines 1′2′3′ divides it in the same anharmonic ratio as 1, 2 or 3 is divided by them: so does the plane 24′ with the lines 1′2′3′: therefore the planes 14′, 24′ are met by 4 in a common point, or, in other words, 4 meets 4′. Further, the point in which each of these lines crosses another is unaltered by deformation: for the relations already established are sufficient to determine definitely the segments of these lines in terms of the segments of the six rods: we can therefore replace the lines by jointed rods.

In the case of the paraboloidal system, in which all the rods of the same series are divided similarly, we have relations of remarkable simplicity. For the orthogonal projection on any plane consists of two series of parallel lines, and the segments of each set of rods are proportional to their projections. By considering the projections on two different planes, the above results follow immediately.

Let us consider now the quasi-cubical system of jointed rods. In the first place, such a system is abundantly possible; for assuming the 9 rods connecting the three layers which lie the same way, and denoting them by the 9 digits, from any point on 1 draw the line which intersects 2 and 3, from the point in which it meets 3 draw the line which intersects 6 and 9, from the point in which it meets 9 draw the line which intersects 8 and 7, and from the point in which it meets 7 draw the line which intersects 4 and 1, as in the diagram. The last line must meet 1 in the point from

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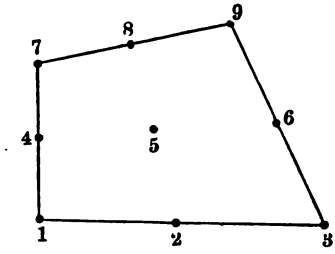
which we started, which gives one condition, and the three other independent circuits in the same layer give three more. Thus the three layers give twelve conditions, which can easily be satisfied by



the nine lines we started with, especially as three of them may be removed by properly choosing the positions of the layers.

Having thus proved the possibility of the arrangement, we proceed as before to count *all* the independent relations of the system, and find whether they are sufficient to fix it absolutely,—or, if not, to find how many modes of deformation it possesses. We project all the independent circuits on the axes, just as before in the case of the binary system. There are 9 binary systems contained in the ternary, 3 sets of 3 each; but it will be clear on consideration that the existence of 2 of these sets determines the third set, which crosses them both. The independent circuits of the ternary system are therefore those of these two sets of binaries, and give equations $6 . 4 . 3$ in number; while the metrical relations of the binaries give $6 . 4$ conditions among the lengths of the segments, which are necessarily included in the former: so that there are $6 . 4 . 2$ or 48 independent equations. There are also 27 relations between the direction cosines of the 27 lines, which are the variables. Thus there are 75 equations in all between these 81 variable direction cosines. But the arbitrary axes introduce into them 3 degrees of indeterminateness. There are therefore still 3 degrees remaining: that is, the jointed system possesses three degrees of internal freedom.

And now the same considerations that we employed in the case of a binary system show that we may introduce any additional number of rods in each set, so that three rods shall meet at each joint, when the system will still possess its three degrees of internal freedom.

This remarkable general result is in agreement with what we can see to be true in particular cases. The simplest case of all is that of a parallelepipedal system formed of three sets of parallel jointed rods: here we can alter all the three angles between the directions of the rods. Another simple case is that of a series of

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equal and similar binary systems (forming paraboloids or hyperboloids) placed one over the other with corresponding joints connected by a third system of parallel rods, whose segments between two of the binary systems are therefore all equal: here the parallel rods have two degrees of freedom, and the binary systems have the third.

The fact that in a binary system 4 segments are determinable in terms of the others shows that such a system is itself determined by two rods crossing three others and jointed to them:—in other words, that a system of confocal hyperboloids is so determined. So also a ternary system is determined by four rods jointed together at different parts of their lengths by three sets of four rods each.

(3) *On some Physiological Experiments. By* W. H. GASKELL, M.D., F.R.S.

(4) *On a method of comparing the concentrations of two solutions of the same substance, but of different strengths. By* A. S. LEA, M.A.

(5) *On the many layered Epidermis of Clivia Nobilis. By* WALTER GARDINER, B.A.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 27, 1884.

ANNUAL GENERAL MEETING.

THE following were elected officers and new members of the Council for the ensuing year:—

President:

Prof. Foster.

Vice-Presidents:

Prof. Stokes, Lord Rayleigh, Prof. Liveing.

Treasurer:

Mr J. W. Clark.

Secretaries:

Mr Trotter, Mr Glazebrook, Mr Vines.

New Members of Council:

Prof. Cayley, Prof. Macalister, Mr Glaisher.

The following communications were made to the Society:

(1) *On the effect of moisture in modifying the refraction of plane polarised light by glass. By* R. T. GLAZEBROOK, M.A.

When plane polarised light is allowed to pass through a wedge or plate of an isotropic transparent medium the relation between the positions of the planes of polarisation in the incident reflected and refracted beams are given by various theories. Brewster, Seebeck, Jamin, Quincke, and others have attempted the verification of these formulae by experiment and the two

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latter have shewn that with most substances between certain values of the angle of incidence the reflected beam is elliptically polarised.

In all such experiments one great difficulty meets us. The effects we are looking for do not seem exactly the same from day to day. Modifications of some nature appear to go on at the surface and the means from results of observations made on different occasions differ by quantities which are large compared with the differences between anyone of the means and the individual observations of the set from which it is deduced.

Thus the following readings represent observations on the position of the plane of polarisation after plane polarised light polarised at an angle of 45° to the plane of incidence has been allowed to pass through a wedge of glass of small angle. These observations were made on six occasions during August last year.

48° 4′	48° 1′
48° 1′·10	47° 56′
47° 58′	47° 58′.20

Each is the mean of five or six observations no two of which probably differed by more than 3′.

The object of the experiments described in the present paper was to determine some of the conditions which produce this variation.

Numerous observations seemed to indicate some connexion between the effect and temperature. Thus readings of the analyser, a Laurent's plate and a Nicol's prism, were found generally to be higher on warm days than on cold.

The following set of observations shew the change on July 19 as the temperature of a thermometer placed almost in contact with the glass rose from 20° to 22°.

Reading.	Temperature.	Time.
		h. m.
32° 21′ 15″	20°.1 C	3 5
24′ 15″	20°.15 „	3 35
25′	21°.8 „	4 5
24′ 45″	21°.8 „	4 35
25′ 15″	22° „	6 40

Now it is improbable that the change is due to inequalities of temperature in the glass which is thin and carefully shaded

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from direct radiation, but still it is difficult to see how change of temperature of the whole, which does not alter the refractive index of the glass, should produce the effect. Moreover the change was not always observed as the temperature varied and some other cause must be sought. The effect of varying the amount of moisture in the air round the glass was then tried.

Dry air was blown from a bellows through a flask filled with Calcium chloride and then through a second with dry cotton-wool on to the surface and damp air was blown through a bottle and then through a second empty bottle to catch the spray.

Observations were made by means of a wet and dry bulb thermometer placed near the glass on the normal hygrometric condition of the air and of the variations produced by the draughts of dry and damp air respectively.

The quantity determined directly is the reading of the analyser circle corresponding to the position of the plane of polarisation of the incident light. As the reading increases the angle between the planes of polarisation of the incident and refracted light is increased and vice versâ.

A large number of observations were made at various angles of incidence with the following results.

Dry air lowers the reading, i.e. brings the planes of polarisation of the incident and refracted waves more nearly into coincidence. Rubbing the surface with dry wash leather lowers the reading which usually comes back after a time to its normal value. The effect of damp air depends on the previous treatment of the surface. If this has not been carefully cleaned with dry leather shortly before the air is blown the damp air *lowers* the reading, but immediately after the blowing is stopped, or in some cases

(1) Condition.	(2) Readings.
Normal	32° 45′
Dry air	32° 41′
Normal	32° 45′
Dry air	32° 41′·30
Normal	32° 44′·30
Damp Air.	
Observation in act of blowing	32° 38′
„ immediately after	32° 48′
„ in act of blowing	32° 37′ 30″
„ immediately after	32° 47′ 30″
„ in act of blowing	32° 37′
„ immediately after	32° 48′

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after it has been continued for some time the reading rises to above its normal value falling back to it in time.

This is shewn in the observations for July 28, the glass having not been rubbed since July 22. These are given in the Table column (1) shewing the conditions under which the readings were taken, and (2) the actual reading. The normal condition means without blowing either dry or damp air on the surface. The angle of incidence was about 45°.

After an interval of an hour the normal reading was 32° 44′.

The temperature changed during the observations from 19° to 20°.

Thus in this case damp air produces, while being blown, an effect of the same sign as the dry but of greater magnitude, while the effect observed just after the blowing has ceased is opposite to that due to dry air.

If however the surface has been rubbed with a dry leather immediately before the damp air is blown the reading is raised at once without the previous lowering.

A very full set of observations taken on August 6 shew the two effects.

Time.	Conditions.	Reading.
h. m.		
1 7	Normal	35° 8′
	Dry air blown	35° 4′ 30″
1 15	Normal	35° 8′ 15″
	Damp air, at first	34° 59′
	then	35° 7′
	and finally	35° 9′ 30″
1 21	Normal	35° 8′ 30″
3 18	Normal	35° 9′ 30″
	Damp air, at first	35° 3′
	then	35° 9′
	and immediately after stopping	35° 13′
4 P.M.	*Glass rubbed.*	
	Normal	35° 8′ 30″
	Damp air blown	35° 15′
	immediately after stopping	35° 15′
	and	35° 12′
4 25	*Glass rubbed.*	
5 23	Normal	35° 8′
	Damp air blown	35° 10′ 30″
5 30	Normal	35° 7′ 30″
	Dry air blown	35° 5′
5 35	Normal	35° 9′

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The angle of incidence here was about 60°, the temperature varied from 23° C. to 24° C. and the tension of aqueous vapour from 11·3 mm. to 14·2 mm. the saturating tension changing from 19·7 mm. to 21·7 mm.

During the first eleven observations the glass had not been cleaned since the previous day, and the effects are the same as previously. After the glass was cleaned the damp air raises the reading without previously lowering it.

Other sets of observations at different angles of incidence gave the same results.

Thus on August 20 we find just after the glass has been cleaned at angle of incidence of 45° the following readings.

Time.	Condition.	Reading.
h. m.		
3 4	Normal	32° 58′ 30″
3 17	Dry air blown	32° 56′
	Normal	32° 58′ 30″
3 25	Damp air blown	32° 59′
3 35	Dry air blown	32° 55′ 30″

There seems too a tendency for the normal readings to rise with e the saturating tension, this is shewn below.

Date.	Value e.	Temperature.	Reading.
Aug. 5, 3 P. M.	11·4 mm.	22°	35° 6′ 30″
„ 3.55	12·1 „	24°	35° 9′ 30″
Aug. 6, 1.7	11·3 „	23°·5	35° 8′
„ 1.21	12·1 „	24°	35° 8′ 30″
„ 5.25	12·7 „	25°	35° 9′

The connexion however is not very marked.

It remains then to account, if possible, for the anomaly in the behaviour of the glass with damp air; but before attempting this it is well to mention that the effects produced by blowing other gases Oxygen, Hydrogen, and Carbonic acid on the surface, were tried and found to be practically inappreciable, while additional experiments made in October, 1884 confirm the results already arrived at as to the effects of dry and damp air.

Observations also were made on the amount of saturation produced; damp and dry air being blown on to the wet bulb.

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The values of e observed on one occasion were:

For Damp air.....19·45 mm.
For Normal state.....11·99 „
For Dry air..... 8·44 „

the saturating tension being 20·66 mm.; thus the air was nearly saturated by the draught of damp air.

The anomalous behaviour of the glass under a draught of moist air may, I think, be explained by some experiments of Magnus, *Pogg Annalen*, vol. CXXI. *Phil. Mag.* Ser. IV., vol. 27.

Magnus found that blowing damp air on to the surface raises its temperature while blowing dry air cools it; the change of temperature being due presumably to slight condensation of moisture on or evaporation from the surface, respectively. I repeated his experiments both with a thermometer and thermopile, and observed the effects. With the thermometer as at first fitted up the effect was very small. Thus there was as the mean of several observations a cooling effect of 0°·03 C., due to dry air, a heating effect of 0°·05 C. due to damp air, while the galvanometer needle was affected about equally in opposite directions owing to the draught. I then cleaned one thermometer bulb *B* carefully and slightly soiled that of another *A*. The two were enclosed in the same glass bulb, and a draught of air blown on to them both.

The readings were as follows:

A.	Dry	25°	24°·9	24°·8	24°·8.
	Damp	25°·4	25°·3	25°·2	25°·2.

B. Dry, Damp } 25° in all cases.

Thus the draughts of air produce no effect on the clean bulb, while damp air heats the soiled one, and dry air cools it.

I then supposed that the draught of damp air might slightly heat the surface on which it was blown, and investigated what the effect of this would be. For this purpose a small spiral of platinum wire was placed near the point of the surface on which the light was incident, and an electric current passed through it. The wire thus being heated raised the temperature of the glass locally, thus straining it and it was found that the analyser readings were lowered at once; and in a very short time the light became so strongly elliptically polarised that it could not be quenched. Thus local heating produces a lowering in the reading; so that if we suppose that when damp air is blown on to a glass surface which has not been recently cleaned that surface is slightly heated the apparently anomalous optical phenomena observed when the draughts of dry air are blown would be accounted for; the glass is heated and the reading lowered.

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We may perhaps account for the slight heating of the soiled surface from the fact that the coating of small dust particles which covers it may form a series of nuclei round which the moisture of the air readily condenses. On the freshly cleaned surfaces there are no such nuclei, the moisture is not condensed and the heating effects are not produced.

The bearing of the results of the experiments on the theory of reflexion must be left for discussion in a future paper.

Postscript. Nov. 1884. The experimental results have been confirmed by further experiments made during the present month.

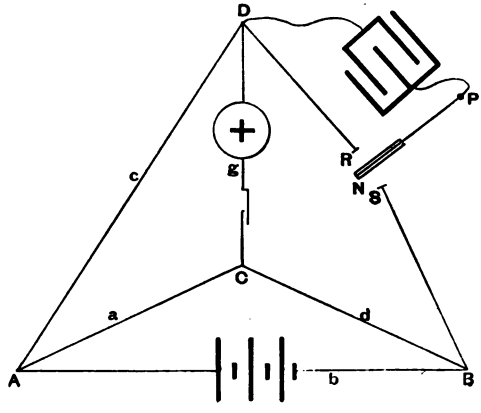
(2) *On some experiments on the measurement of the capacity of a condenser.* *By* L. R. WILBERFORCE, B.A.

The method employed in the following experimental determination was the same as that used by J. J. Thomson in his measurement of "*v*", (*Phil. Trans.* iii. 1883), and may be thus briefly described.

For the resistance in one arm of a Wheatstone's bridge there is substituted a broken circuit, a condenser and a commutator which connects the electrodes of the condenser alternately with the broken ends of the circuit and with each other.

The commutator is worked at such a rate that its periodic time is small compared with the time of swing of the galvanometer needle. A key is introduced into the galvanometer circuit, and the resistances are adjusted so that the position of equilibrium of the needle is the same when the circuit is open and when it is closed.

If this is the case, a relation will exist between the periodic



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time of the commutator, the resistances of the bridge, and the capacity of the condenser.

The arrangement may be represented by a slight modification of the ordinary Wheatstone bridge diagram. The commutator consists of the spring PN which is fixed at P, and which strikes R and S alternately, making (n) double vibrations a second. The vibrations are regulated and maintained by a current rendered intermittent by passing through a fork interrupter.

If we neglect all the coefficients of self- and mutual-induction of the resistance coils, and the resistances DR, SB, the relation between C, the capacity, n, and the resistances has been investigated by Thomson, and is

$$nC = \frac{a}{cd} \cdot \frac{\left[1 - \frac{a^2}{(a+c+g)(a+b+d)}\right]}{\left[1 + \frac{ab}{c(a+b+d)}\right]\left[1 + \frac{ag}{d(a+c+g)}\right]}.$$

The experiments were all made at the Cavendish Laboratory during July and August, 1884.

The condenser used was divided into four parts whose reputed capacities were ·1, ·2, ·3, ·4, of a microfarad respectively.

The galvanometer used was a reflecting one with a resistance of 11,000 B.A. units. A vertical slit was placed in front of the lamp used so that a narrow band of light with very well defined edges was reflected on to the scale. The readings were taken for the left-hand edge and were correct to one scale division, the divisions being a millimetre apart.

The fork-interrupters used had frequencies (N_{32} and N_{64}) of about 32 and 64 double vibrations per second respectively. They will be referred to as the "32" and "64" forks. A battery of three pint Groves was used for driving them, while in the bridge three Léclanché cells were used.

The resistances (a) and (d) were coils of 10 and 1000 B.A. units, while (c) was a resistance box by means of which we could proceed by single units up to 1000.

Every precaution was taken to make the resistances of connection extremely small. The key used for opening and closing the galvanometer circuit consisted of two wires whose ends dipped into four mercury cups. It was arranged that when the galvanometer was not in circuit the ends of its coil should be connected.

Various precautions instantly suggest themselves, which must be taken in order that:

(i) the conditions assumed in the investigation of our formula should be the conditions of the experiment;

(ii) the calculated value of C should be correct within as small a fraction of itself as possible;

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(iii) the observation can be most conveniently made;
(iv) the calculations can be most conveniently effected.

Under the first head we notice that (n) must not be so great that the time of contact between P, and either S or R is too small to allow of the condenser being fully charged or discharged.

Again we must be careful that the resistances BSP when P and S are in contact and DRP when P and R are in contact are negligible.

That this should be the case it is necessary that the contact resistances between P and R and between P and S should be very small. If P strikes R and S hard a better contact will be made than if it strikes gently, and thus we will get better results, the stiffer our spring and the greater the power of our driving battery, both for the above reason, and because in that case we diminish the time during which P is in contact with neither S nor R, and thus we might increase (n) and still have our condenser fully charged and discharged at each contact.

Again we must be careful that (n) is not so small as to allow the phenomena of electric absorption and residual discharge to be produced, for in that case we should get too large a value for C.

Under the second head we notice that the greater the electromotive force of our battery, the more sensitive our galvanometer, and the greater the value of (n), the more delicate the adjustment necessary to produce a balance, and consequently the more accurate the method.

Under the third head we notice that, since the observations can only be conveniently taken when the needle is at rest, it should be made as steady as possible. We must particularly guard against its hanging in too weak a field of force, for the effect of the transient currents through the galvanometer is to diminish the stability of the needle, as has been proved by Lord Rayleigh.

Under the fourth head we notice that, our formula being:

$$nC = \frac{a}{cd}\left[1 - \frac{a^2}{(a+c+g)(a+b+d)}\right] \cdot \left[1 + \frac{ab}{c\,(a+b+d)}\right]^{-1}\left[1 + \frac{ag}{d\,(a+c+g)}\right]^{-1},$$

and g being large and a and b being small compared with the other resistances, it becomes

$$nC = \frac{a}{cd}\left(1 - \frac{a}{d}\right).$$

Hence we see that it is more convenient that (c) should be the adjustable resistance, for then the factor $\left(1 - \frac{a}{d}\right)$ is a constant one. The advantages of this method have been exhaustively discussed

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by Glazebrook (*Phil. Mag.*, Aug. 1884). He has also mentioned some of its uncertainties. One of its principal disadvantages appears to me to be the fact that though it is a null method inasmuch as the equilibrium position of the needle is not to be disturbed by the passage of the transient currents through the galvanometer coils, yet it is not so inasmuch as the needle instead of never moving from this position is always jerked aside when the currents either begin or cease to pass, and hence, instead of observing an absence of effect we have to measure the positions of a spot of light at two times, and determine whether they are the same. This initial jerk is due to the fact that the transient and constant current do not balance at every moment. and that their average effect requires some time to show itself. Thus this disadvantage is inherent to the method.

The observations are thus not only rendered more difficult, but a considerable element of uncertainty is introduced as to whether the zero point of the needle may not have changed between the measurements.

We may take for our zero point the mean of the points at which the needle rests before and after the currents pass, but this will not be satisfactory if the change of zero is at all rapid, as it sometimes was.

I had attempted three months previously to make some determinations by this method but rejected them as the results obtained were untrustworthy owing to the unsteadiness of the galvanometer needle, and also hopelessly inconsistent among themselves.

The unsteadiness was partly due to the effects of draughts which were afterwards excluded by enclosing the galvanometer in a wooden box with a glass shutter, partly to the weakness of the field of force in which the needle hung, which was also remedied, while the inconsistency and a great deal of the unsteadiness was caused by leakage due to imperfect insulation. After this was discovered, the greatest precautions were taken to secure good insulation, wires covered with gutta-percha being used for all the connections, and all the apparatus used being supported by pieces of glass coated with paraffin.

For our calculations of the capacities it was still necessary to know as accurately as possible the rates of the forks used.

The "32" and "64" forks were compared directly. It was then discovered that the "32" fork was wrongly weighted, so as to vibrate about $32\frac{1}{2}$ times a second. As this would prove troublesome when it was compared with the standard, it was reweighted and compared afresh with the "64" fork. The fork as reweighted, which we shall call the $\boxed{32}$ fork, was then made to drive a fork of about four times its frequency, and this latter was compared

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with the standard fork (whose frequency was determined by Lord Rayleigh) by counting the beats between the two. A resonator being used, this could be very conveniently and accurately done.

To compare the "32" or the [32] and the "64" forks both were set vibrating, and the dipper carried by the latter was viewed over the top of the former. A screen was brought down in front of the prong of the fork over which we looked so that we could only see the dipper between the screen and the prong when the latter was very near its lowest position.

A narrow horizontal slit ruled on a plate of smoked glass was sometimes used instead of a screen. The dipper was thus seen in a state of slow vibration, and by placing behind it a piece of paper with a line ruled on it, which was just reached by the dipper in its highest position, the vibrations could be very accurately counted even when extremely slow.

The following is a specimen of the electromagnetic observations. The extreme columns give the resistance (c) in B.A. units, and the equilibrium position of the left edge of the band of light on the scale when the currents are passing through the galvanometer. We write c_1 for the capacity of the condenser marked ·1.

August 13*th*.

$n = N_{32}$, $C = c_1 + c_2 + c_3$.

Resistance (c).	Zero point.	Equilibrium position.
	56	
510		31
	56	
	55	
520		51
	55	
	55	
530		71
	54	
	55	
522		55
	$55\frac{1}{2}$	
	$55\frac{1}{2}$	
522		$55\frac{1}{2}$
	54	
	55	
522		57
	57	
	58	
	55	
522		56

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Resistance (*c*).	Zero point.	Equilibrium position.
	56	
	57	
	58½	
	59	
522		60
	59	
	57	
521		53
	54	
	54	
521		54½
	56	
	56	
521		57
	58	
	59	
521		58
	59	
	58	
522		61
	60	
522		61
	60	

From this series of observations we take 521·5 for our value of *c*.

We notice that a difference of one unit in (*c*) makes a difference of two scale divisions in the equilibrium position, and this is about the degree of sensitiveness that we found throughout.

The following is a specimen of the observations taken in comparing the forks.

August 18*th*.

"32" and "64" compared.

Roughly we see 49 or 50 double vibrations of the dipper per minute.

Counting with the chronometer, we get

49 vibs. in 60½ secs.
49 vibs. in 60 secs.
49 vibs. in 61 secs.

"32" weighted, 49 vibs. in 71½ secs.

"32" more weighted,
25 vibs. in 67½ secs.

"32" as at first, 49 vibs. in 60½ secs.

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Mean time of a vibration $= \frac{121}{98}$ sec.

Thus $$2N_{32} \pm \frac{98}{121} = N_{64}.$$

The lower sign is the one to be taken.

Putting N_t for the frequency of the standard fork, we get in the same way the equations:

$$2N_{[32]} + \frac{1}{122} = N_{64},$$

$$4N_{[32]} - \frac{19}{60} = N_t \text{ (temperature} = 20^\circ \text{ C.)}.$$

Now (Rayleigh, *Phil. Trans.* I. 1883)

$$N_t = 128{\cdot}140\,\{1 - (t-16) \times {\cdot}00011\}.$$

Hence
$$N_t = 128{\cdot}0836,$$
$$N_{[32]} = 32{\cdot}10005,$$
$$N_{64} = 64{\cdot}2083,$$
$$N_{32} = 32{\cdot}5091.$$

Substituting our resistances in Thomson's formula, we have

$$nC = \frac{a}{cd}\left(1 + \frac{1}{100}\right)^{-1}, \text{ nearly,}$$
$$= \frac{99}{100}\,\frac{a}{cd},$$

the resistances being in C.G.S. units.

Now the B.A. unit is, as determined by Lord Rayleigh,

$${\cdot}987 \times 10^9.$$

Hence, if our resistances are in B.A. units, we have

$$C = \frac{1}{n} \cdot \frac{99}{100}\,\frac{1}{{\cdot}987 \times 10^9} \cdot \frac{a}{cd}.$$

The results obtained are given in the following table:

Condenser used.	(n)	(c)	Capacity.
$c_1 + c_2 + c_3$	32·509	521·5	$\cdot5918 \times 10^{-15}$,
$c_2 + c_4$	32·509	521·5	$\cdot5918 \times 10^{-15}$,
$c_1 + c_2$	64·208	526	$\cdot2969 \times 10^{-15}$,
c_3	64·208	526	$\cdot2969 \times 10^{-15}$,
c_2	64·208	789·5	$\cdot1979 \times 10^{-15}$,
c_4	64·208	395·5	$\cdot3949 \times 10^{-15}$,
$c_1 + c_3$	64·208	395	$\cdot3955 \times 10^{-15}$.

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If we examine these results we see that from the last five we can deduce c_1 in two ways, and that we shall get for our results:

$$\cdot 0990 \times 10^{-15},$$

and $$\cdot 0986 \times 10^{-15}.$$

Again, from them we get,

$$c_1 + c_2 + c_3 = \cdot 5938 \times 10^{-15},$$
$$c_2 + c_4 = \cdot 5928 \times 10^{-15},$$
$$c_1 + c_3 + c_2 = \cdot 5934 \times 10^{-15}.$$

Thus we see that all our results agree fairly well among themselves.

November 10, 1884.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows:

T. C. T. Reeve, B.A., Caius College.
H. M. Elder, B.A., Trinity College.

The following communications were made to the Society:

(1) *On some experiments on the Liver ferment.* By Miss F. EVES, communicated by Mr A. S. LEA.

The experiments on which this communication are based, were made with the object of obtaining some more definite information as to the existence and action of the ferment which in the liver is usually supposed to be the chief factor in the conversion of glycogen into sugar.

From general consideration it appears probable that the conversion of glycogen into sugar in the liver may be regarded as due rather to the special activity of the hepatic protoplasm than to the action of any specific ferment. An examination of the literature on the nature of the ferment shews that an amylolitic ferment can be obtained from the liver; but it shews also some uncertainty as to its preparation by several observers, and that it is not so characteristically present in this organ as the importance of the part it plays would seem to imply. No statements exist as to the nature of the product which can be obtained by the action of this ferment on starch and glycogen, since it does not seem to have been isolated in a sufficiently pure state to allow of such observations being made.

An effort was made in these experiments to obtain the supposed ferment in a fairly pure state and this was successfully done

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as far as it is possible to consider that such bodies can be isolated. It appears that an amylolytic ferment can be prepared from the liver by the method adopted, but that the quantity, as judged by its activity, is less than might be expected if it plays any very specific part in this organ, and is in fact not greater than the amount which can be obtained from nearly every tissue and fluid of the body.

The action of the ferment in as concentrated an extract as could be prepared was tested on starch, and it was found that though the sugar formed could not be proved to be definitely maltose, it was very certainly not dextrose. This is important since the sugar found in the liver itself post mortem is stated by all observers to be dextrose. It is hence still more probable that the ferment extracted is only the ordinary amylolytic ferment met with generally throughout the tissues.

(2) *On the supra renal bodies.* By Mr W. F. R. WELDON.

(3) *On the supposed presence of protoplasm in the intercellular spaces.* By Mr W. GARDINER.

Russow's discovery of the existence of a so-called "intercellular protoplasm" having been confirmed by several other investigators, the author was led to examine the whole matter with the greatest possible detail from a comparative and developmental point of view, since the results were directly opposed to the views which he had previously stated in his paper "On the constitution of the cell-wall and middle-lamella." He finds that all his experiments unanimously point to the fact that the substance present is not protoplasm, but mucilage, and that the mucilaginous degeneration of the external layers of the cell-wall is a phenomenon of almost universal occurrence.

(4) *On a proteid occurring in plants.* By Mr J. R. GREEN.

The proteid which was the subject of the paper was described as possessing properties similar in some respects to those of the peptones and in others to those of the globulins. It is allied to the former group by being soluble in distilled water, the reaction of the solution being neutral; in not being coagulated on boiling; and in being capable of dialysis. It resembles the globulins in being precipitated by saturation of its solution by neutral salts such as magnesium sulphate, or by the passage through it of a stream of carbonic acid gas. It has been found to occur in Manihot glaziovii, Mimusops globosa and a few other plants.

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(5) *On the secretory hairs on the stem of Thunbergia laurifolia.* By Messrs. W. GARDINER and R. I. LYNCH.

In this investigation Mr Lynch made the observations as to the external morphology of the hairs, and Mr Gardiner examined their microscopic structure and development. The authors stated that the hairs exhibited many points of interest. They occurred on the stem only at the base of each leaf insertion, and were thus localised in a very exceptional manner. They were of a most perfect cup form, and the watery and slightly acid secretion appeared on the free surface and did not cause the raising of the cuticle as so usually occurs in most secretory structures. They are apparently unique as to their distribution, since they neither occur in Thunbergia alata nor in such closely related genera as Meyenia and Hexacentris. They are essentially transitory in nature and are developed from a single epidermal cell.

November 24, 1884.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:

(1) *On the developments of K', E', J', G' in powers of the modulus.* By J. W. L. GLAISHER, M.A.

The quantities E, J, G, § 1.

§ 1. Besides the quantities K and E of Legendre and Jacobi and the quantity $J=K-E$ of Weierstrass I have found it desirable to treat also $E-k'^2K$ as a fundamental quantity, forming one of a triad of which E and J are the other two members*. Denoting $E-k'^2K$ by G it is found that not only E, J, G but also $J-G$, $E+G$, $E-J$ enter into the formulæ of Elliptic Functions in such a manner that these six functions form a special group by themselves.

Using accented letters to denote the same functions of k' that the unaccented letters are of k we are thus concerned with the following fourteen functions:

* See a paper "On the Quantities K, E, J, G, K', E', J', G' in Elliptic Functions," *Quarterly Journal of Mathematics*, Vol. xx., pp. 313—361.

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$$\begin{aligned}
&K, && K',\\
&E, && E',\\
&J = K - E, && J' = K' - E',\\
&G = E - k'^2 K, && G' = E' - k^2 K',\\
&J - G = (1 + k'^2) K - 2E, && J' - G' = (1 + k^2) K' - 2E',\\
&E + G = 2E - k'^2 K, && E' + G' = 2E' - k^2 K',\\
&E - J = 2E - K, && E' - J' = 2E' - K'.
\end{aligned}$$

Formulæ for E, J, G, &c. in terms of k, § 2.

§ 2. The following formulæ give the expansions of $\frac{2K}{\pi}$, $\frac{2E}{\pi}$, &c. in ascending powers of k^2:

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2 . 3^2}{2^2 . 4^2} k^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} k^6 + \&c.,$$

$$\frac{2E}{\pi} = 1 - \frac{1}{2^2} k^2 - \frac{1^2 . 3}{2^2 . 4^2} k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2} k^6 - \&c.,$$

$$\frac{2J}{\pi} = \frac{1}{2} k^2 + \frac{1^2 . 3}{2^2 . 4} k^4 + \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6} k^6 + \&c.,$$

$$\frac{2G}{\pi} = \frac{1}{2} k^2 + \frac{1^2}{2^2 . 4} k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6} k^6 + \&c.,$$

$$\frac{2(J-G)}{\pi} = \frac{1^2}{2 . 4} k^4 + \frac{1^2 . 3^2}{2^2 . 4 . 6} k^6 + \&c.,$$

$$\frac{2(E+G)}{\pi} = 1 + \frac{1}{2^2} k^2 + \frac{1^2}{2^2 . 4^2} k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6^2} k^6 + \&c.,$$

$$\frac{2(E-J)}{\pi} = 1 - \frac{1 . 3}{2^2} k^2 - \frac{1^2 . 3 . 5}{2^2 . 4^2} k^4 - \frac{1^2 . 3^2 . 5 . 7}{2^2 . 4^2 . 6^2} k^6 - \&c.$$

The formulæ for $\frac{2J}{\pi}$ and $\frac{2G}{\pi}$ may be deduced from those for $\frac{2K}{\pi}$ and $\frac{2E}{\pi}$, either by means of the algebraical formulæ

$$J = K - E, \qquad G = E - (1 - k^2) K,$$

or by means of the differential formulæ

$$J = -k \frac{dE}{dk}, \qquad G = k(1 - k^2) \frac{dK}{dk}.$$

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Formulæ for K', E', &c. in terms of k, § 3.

§ 3. The formulæ to which the present paper principally relates are those giving the expansions of K', E', &c. in ascending powers of k^2.

I have found that these developments may be written in the following form:

$$K' = \log\frac{4}{k} + \frac{1^2}{2^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2}\right)k^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4}\right)k^4 + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6}\right)k^6 + \&c.$$

$$E' = 1 + \frac{1}{2}\left(\log\frac{4}{k} - \frac{1}{1} + \frac{1}{2}\right)k^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{1}{3} + \frac{1}{4}\right)k^4 + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{1}{5} + \frac{1}{6}\right)k^6 + \&c.$$

$$J' = \log\frac{4}{k} - 1 - \frac{1}{2^2}\left(\log\frac{4}{k} - \frac{1}{1} + \frac{2}{2}\right)k^2 - \frac{1^2\cdot 3}{2^2\cdot 4^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{1}{3} + \frac{2}{4}\right)k^4 - \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{1}{5} + \frac{2}{6}\right)k^6 - \&c.$$

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$$G' = 1$$
$$- \frac{1^0}{2}\left(\log\frac{4}{k} - \frac{0}{1} + \frac{1}{2}\right)k^2$$
$$- \frac{1^2}{2^2 \cdot 4}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} + \frac{1}{4}\right)k^4$$
$$- \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} + \frac{1}{6}\right)k^6$$
$$- \&c.$$

$$J' - G' = \log\frac{4}{k} - 2$$
$$+ \frac{1^0}{2^2}\left(\log\frac{4}{k} - \frac{0}{1} + \frac{2}{2}\right)k^2$$
$$+ \frac{1^2}{2^2 \cdot 4^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} + \frac{2}{4}\right)k^4$$
$$+ \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} + \frac{2}{6}\right)k^6$$
$$+ \&c.$$

$$E' + G' = 2$$
$$- \tfrac{1}{2}k^2$$
$$+ \frac{1^2}{2 \cdot 4}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{1}{2} + \frac{1}{4}\right)k^4$$
$$+ \frac{1^2 \cdot 3^2}{2^2 \cdot 4 \cdot 6}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{6}\right)k^6$$
$$+ \&c.$$

$$E' - J' = -\log\frac{4}{k} + 2$$
$$+ \frac{1 \cdot 3}{2^2}\left(\log\frac{4}{k} - \frac{1}{1} + \frac{2}{2} - \frac{1}{3}\right)k^2$$
$$+ \frac{1^2 \cdot 3 \cdot 5}{2^2 \cdot 4^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{1}{3} + \frac{2}{4} - \frac{1}{5}\right)k^4$$
$$+ \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2}\left(\log\frac{4}{k} - \frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{1}{5} + \frac{2}{6} - \frac{1}{7}\right)k^6$$
$$+ \&c.$$

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§ 4. The coefficient of k^{2n} in each of these seven expressions is of the form

$$\pm r\left(\log\frac{4}{k} - R\right),$$

where r is a fraction whose numerator and denominator are products of factors, and R is a numerical quantity, the connexion of which with r is best exhibited by assigning to n a particular value. Taking, for example, $n = 3$, the values of r are

(i) K', $\dfrac{1^2.3^2.5^2}{2^2.4^2.6^2}$,

(ii) E', $\dfrac{1^2.3^2.5}{2^2.4^2.6}$,

(iii) J', $\dfrac{1^2.3^2.5}{2^2.4^2.6^2}$,

(iv) G', $\dfrac{1^2.3^2}{2^2.4^2.6}$,

(v) $J'-G'$, $\dfrac{1^2.3^2}{2^2.4^2.6^2}$,

(vi) $E'+G'$, $\dfrac{1^2.3^2}{2^2.4.6}$,

(vii) $E'-J'$, $\dfrac{1^2.3^2.5.7}{2^2.4^2.6^2}$,

and the corresponding values of R are

(i) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6}$,

(ii) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{1}{5} - \frac{1}{6}$,

(iii) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} - \frac{1}{5} - \frac{2}{6}$,

(iv) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} - \frac{1}{6}$,

(v) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} - \frac{2}{6}$,

(vi) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{1}{4} - \frac{1}{6}$,

(vii) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7}$.

In all cases the quantities r and R are so connected that to every factor m^μ in r there corresponds a term $\pm\frac{\mu}{m}$ in R, the sign being positive or negative according as m^μ occurs in a numerator or denominator.

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The law of the coefficients, §§ 5, 6.

§ 5. With respect to this curious law the following theorems may be noticed:

(i) if the law holds good in the case of two series in which the factors multiplying $k^{2m} \log \frac{4}{k}$ are r and $r\alpha$ respectively, it will hold good in the case of the sum and difference of the two series, for

$$r\alpha\left(R+\frac{1}{\alpha}\right)+rR=r(\alpha+1)\left(R+\frac{1}{\alpha+1}\right),$$

and

$$r\alpha\left(R+\frac{1}{\alpha}\right)-rR=r(\alpha-1)\left(R+\frac{1}{\alpha-1}\right).$$

The case in which the corresponding multipliers are r and $r\frac{1}{\beta}$ is obviously identical with that just considered, but the independent proof is worth noticing, viz.

$$rR+\frac{r}{\beta}\left(R-\frac{1}{\beta}\right)=r\frac{\beta+1}{\beta}\left(R+\frac{1}{\beta+1}-\frac{1}{\beta}\right),$$

$$rR-\frac{r}{\beta}\left(R-\frac{1}{\beta}\right)=r\frac{\beta-1}{\beta}\left(R+\frac{1}{\beta-1}-\frac{1}{\beta}\right);$$

(ii) if the law holds good for two series in which the corresponding factors are r and $r\frac{\alpha}{\beta}$, it will hold good for their difference, if $\beta=\alpha\pm1$, for

$$rR-r\frac{\alpha}{\beta}\left(R+\frac{1}{\alpha}-\frac{1}{\beta}\right)=r\frac{\beta-\alpha}{\beta}\left(R-\frac{1}{\beta}\right);$$

(iii) if the law holds good for two series in which the corresponding factors are r and $r\frac{(\alpha-1)(\alpha+1)}{\alpha^2}$ it will hold good also for their difference, for

$$rR-r\frac{(\alpha-1)(\alpha+1)}{\alpha^2}\left(R+\frac{1}{\alpha-1}+\frac{1}{\alpha+1}-\frac{2}{\alpha}\right)$$
$$=r\frac{1}{\alpha^2}\left(R-\frac{2}{\alpha}\right).$$

It is obvious that in this theorem we may suppose the corresponding factors to be r and $r\frac{\alpha^2}{(\alpha-1)(\alpha+1)}$.

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(iv) If the law holds good in the case of any series, it will hold good also for its derivative, for

$$k\frac{d}{dk}r\left(\log\frac{4}{k}-R\right)k^{2n}=r\,.\,2n\left(\log\frac{4}{k}-R\right)k^{2n}-rk^{2n}$$

$$=r\,.\,2n\left(\log\frac{4}{k}-R-\frac{1}{2n}\right)k^{2n}.$$

§ 6. If we start with the series for K' and E' in both of which the law holds good, we may deduce the series for J' and G' by means of the formulæ,

$$J'=K'-E',$$

$$G'=E'-k^2K',$$

respectively. The series for J' and G', derived from those for K' and E' by these equations, afford examples of the application of theorem (ii). The series for $J'-G'$, derived from those for J' and G', affords an example of (ii), and the series for $E'+G'$ and $E'-J'$, derived from those for E', J', G', afford examples of (i).

The series for K' and E' may be derived from those for G' and J' by means of the equations

$$K'=-\frac{1}{k}\frac{dG'}{dk},\qquad E'=-k\frac{dJ'}{dk};$$

the series so derived afford examples of (iv).

The series for J' and G' may be derived from those for E' and K' by means of the equations

$$J'=\frac{1}{k}(1-k^2)\frac{dE'}{dk},\qquad G'=-k(1-k^2)\frac{dK'}{dk};$$

the series for J' and G' so derived afford examples of (iii) and (iv) combined.

The theorems (i), (ii), (iii), (iv) serve to explain the occurrence of the law in certain of the series, as derived from others; but it is none the less remarkable on this account that the law should hold good in all the seven series, as certain restricted conditions with respect to the coefficients have to be satisfied in order that the theorems may admit of application.

The Notation, § 7.

§ 7. The meanings assigned to J and J' by Weierstrass* were $J=K-E$, $J'=E'$. Thus Weierstrass used J' in place of E'. As however it is convenient always to denote by accented letters the same functions of k' that the unaccented letters are of k, I have retained E' and used J' to denote $K'-E'$.

* *Crelle's Journal*, Vol. LII. p. 361.

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I found it advantageous in working with the quantities E, J, G to use, instead of J, the quantity $-J$. Putting therefore $I=-J$ the three quantities considered are E, I, G.

It was found to be convenient also to denote by separate letters the three quantities $\frac{1}{2}(I+G)$, $\frac{1}{2}(G+E)$, $\frac{1}{2}(E+I)$. Denoting them by U, V, W we consider therefore the six quantities E, I, G, U, V, W, defined by the equations

$$I = E - K,$$
$$G = E - k'^2 K,$$
$$U = \tfrac{1}{2}(I+G),$$
$$V = \tfrac{1}{2}(G+E),$$
$$W = \tfrac{1}{2}(E+I).$$

The six quantities E, I, G, U, V, W, § 8.

§ 8. Expressing the six quantities in terms of E and K, and of I and K, we have

$$\begin{aligned}
E &= E && = I+K,\\
I &= E-K && = I,\\
G &= E-k'^2K && = I+k^2K,\\
U &= E-\tfrac{1}{2}(1+k'^2)K && = I+\tfrac{1}{2}k^2K,\\
V &= E-\tfrac{1}{2}k'^2K && = I+\tfrac{1}{2}(1+k^2)K,\\
W &= E-\tfrac{1}{2}K && = I+\tfrac{1}{2}K,
\end{aligned}$$

the six quantities E', I', G', U', V', W' denoting the same functions of k' that E, I, G, U, V, W are of k.

Systems of formulæ for K, E, I, G, &c. and K', E', I', G', &c., §§ 9—11.

§ 9. The systems of formulæ given in §§ 2 and 3 may be expressed uniformly in the following manner.

Let
$$K_1 = \frac{2K}{\pi},$$
$$E_1 = \frac{2E}{\pi}, \qquad U_1 = \frac{2U}{\pi},$$
$$I_1 = \frac{2I}{\pi}, \qquad V_1 = \frac{2V}{\pi},$$
$$G_1 = \frac{2G}{\pi}, \qquad W_1 = \frac{2W}{\pi},$$

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then, by § 2,

$$K_1 = 1 + \frac{1^2}{2^2}k^2 + \frac{1^2.3^2}{2^2.4^2}k^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}k^6 + \frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2}k^8 + \&c.,$$

$$E_1 = 1 - \frac{1}{2^2}k^2 - \frac{1^2.3}{2^2.4^2}k^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}k^6 - \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2}k^8 - \&c.,$$

$$I_1 = -\frac{1}{2}k^2 - \frac{1^2.3}{2^2.4}k^4 - \frac{1^2.3^2.5}{2^2.4^2.6}k^6 - \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8}k^8 - \&c.,$$

$$G_1 = \frac{1^0}{2}k^2 + \frac{1^2}{2^2.4}k^4 + \frac{1^2.3^2}{2^2.4^2.6}k^6 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2.8}k^8 + \&c.,$$

$$2U_1 = -\frac{1^2}{2.4}k^4 - \frac{1^2.3^2}{2^2.4.6}k^6 - \frac{1^2.3^2.5^2}{2^2.4^2.6^2.8}k^8 - \&c.,$$

$$2V_1 = 1 + \frac{1^0}{2^2}k^2 + \frac{1^2}{2^2.4^2}k^4 + \frac{1^2.3^2}{2^2.4^2.6^2}k^6 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2.8^2}k^8 + \&c.,$$

$$2W_1 = 1 - \frac{1.3}{2^2}k^2 - \frac{1^2.3.5}{2^2.4^2}k^4 - \frac{1^2.3^2.5.7}{2^2.4^2.6^2}k^6 - \&c.$$

Let K_2, E_2, I_2, G_2, $2U_2$, $2V_2$, $2W_2$ be seven quantities defined by the following series, in which the coefficients of the terms are connected with the coefficients in the above series for K_1, E_1, I_1, G_1, $2U_1$, $2V_1$, $2W_1$ by the law explained in § 4.

$$K_2 = \frac{1^2}{2^2}\left(\tfrac{2}{1} - \tfrac{2}{2}\right)k^2 + \frac{1^2.3^2}{2^2.4^2}\left(\tfrac{2}{1} - \tfrac{2}{2} + \tfrac{2}{3} - \tfrac{2}{4}\right)k^4 + \&c.,$$

$$E_2 = 1 - \frac{1}{2^2}\left(\tfrac{1}{1} - \tfrac{2}{2}\right)k^2 - \frac{1^2.3}{2^2.4^2}\left(\tfrac{2}{1} - \tfrac{2}{2} + \tfrac{1}{3} - \tfrac{2}{4}\right)k^4 - \&c.,$$

$$I_2 = 1 - \frac{1}{2}\left(\tfrac{1}{1} - \tfrac{1}{2}\right)k^2 - \frac{1^2.3}{2^2.4}\left(\tfrac{2}{1} - \tfrac{2}{2} + \tfrac{1}{3} - \tfrac{1}{4}\right)k^4 - \&c.,$$

$$G_2 = 1 + \frac{1^0}{2}\left(\tfrac{0}{1} - \tfrac{1}{2}\right)k^2 + \frac{1^2}{2^2.4}\left(\tfrac{2}{1} - \tfrac{2}{2} - \tfrac{1}{4}\right)k^4 - \&c.,$$

$$2U_2 = 2 - \tfrac{1}{2}k^2 - \frac{1^2}{2.4}\left(\tfrac{2}{1} - \tfrac{1}{2} - \tfrac{1}{4}\right)k^4 - \&c.,$$

$$2V_2 = 2 + \frac{1^0}{2^2}\left(\tfrac{0}{1} - \tfrac{2}{2}\right)k^2 + \frac{1^2}{2^2.4^2}\left(\tfrac{2}{1} - \tfrac{2}{2} - \tfrac{2}{4}\right)k^4 - \&c.,$$

$$2W_2 = 2 - \frac{1.3}{2^2}\left(\tfrac{1}{1} - \tfrac{2}{2} + \tfrac{1}{3}\right)k^2 - \frac{1^2.3.5}{2^2.4^2}\left(\tfrac{2}{1} - \tfrac{2}{2} + \tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5}\right)k^4 - \&c.$$

In all the seven series, the law noticed in § 4 is followed in the formation of the coefficients of k^4 and higher powers of k^2. It is followed also by the coefficients of k^2, except in the case of $2U_2$, and, with reference to this case of exception, it is to be observed that the series for $2U_1$, from which that for $2U_2$ is derived, begins with the term in k^4, and that the law of the

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factors in $2U_1$ is such that they cannot be continued backwards one term.

The quantities U_2, V_2, W_2 are such that

$$U_2 = \tfrac{1}{2}(I_2 + G_2),$$
$$V_2 = \tfrac{1}{2}(G_2 + E_2),$$
$$W_2 = \tfrac{1}{2}(E_2 + I_2).$$

§ 10. Denoting by K_1, E_1, I_1, &c. and K_2, E_2, I_2 ... the series to which they are equated in the preceding section, the expansions of K', E', I', &c. in powers of k^2 are given by the equations:

$$K' = K_1 \log \frac{4}{k} - K_2,$$

$$E' = -I_1 \log \frac{4}{k} + I_2,$$

$$I' = -E_1 \log \frac{4}{k} + E_2,$$

$$G' = -G_1 \log \frac{4}{k} + G_2,$$

$$U' = -V_1 \log \frac{4}{k} + V_2,$$

$$V' = -U_1 \log \frac{4}{k} + U_2,$$

$$W' = -W_1 \log \frac{4}{k} + W_2.$$

§ 11. The values of K', E', I', &c. in terms of K, E, I, &c. and the series K_2, E_2, I_2, &c. are given by the equations:

$$K' = \frac{2K}{\pi} \log \frac{4}{k} - K_2,$$

$$E' = -\frac{2I}{\pi} \log \frac{4}{k} + I_2,$$

$$I' = -\frac{2E}{\pi} \log \frac{4}{k} + E_2,$$

$$G' = -\frac{2G}{\pi} \log \frac{4}{k} + G_2,$$

$$U' = -\frac{2V}{\pi} \log \frac{4}{k} + V_2,$$

$$V' = -\frac{2U}{\pi} \log \frac{4}{k} + U_2,$$

$$W' = -\frac{2W}{\pi} \log \frac{4}{k} + W_2.$$

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The formulæ in the case when k is small, §§ 12—15.

§ 12. When k is small,

$$\frac{2K}{\pi} = K_1 = 1 + \tfrac{1}{4}k^2,$$
$$\frac{2E}{\pi} = E_1 = 1 - \tfrac{1}{4}k^2,$$
$$\frac{2I}{\pi} = I_1 = -\tfrac{1}{2}k^2,$$
$$\frac{2G}{\pi} = G_1 = \tfrac{1}{2}k^2,$$
$$\frac{2U}{\pi} = U_1 = -\tfrac{1}{16}k^4,$$
$$\frac{2V}{\pi} = V_1 = \tfrac{1}{2} + \tfrac{1}{8}k^2,$$
$$\frac{2W}{\pi} = W_1 = \tfrac{1}{2} - \tfrac{3}{8}k^2,$$

and

$$K_2 = \tfrac{1}{4}k^2,$$
$$E_2 = 1 - \tfrac{5}{128}k^4,$$
$$I_2 = 1 - \tfrac{1}{4}k^2,$$
$$G_2 = 1 - \tfrac{1}{4}k^2,$$
$$U_2 = 1 - \tfrac{1}{4}k^2,$$
$$V_2 = 1 - \tfrac{1}{8}k^2,$$
$$W_2 = 1 - \tfrac{1}{8}k^2.$$

§ 13. Thus, if we neglect terms of the order k^2 in the series K_2, E_2, &c., we have

$$K' = \frac{2K}{\pi}\log\frac{4}{k},$$
$$E' = -\frac{2I}{\pi}\log\frac{4}{k} + 1,$$
$$I' = -\frac{2E}{\pi}\log\frac{4}{k} + 1,$$
$$G' = -\frac{2G}{\pi}\log\frac{4}{k} + 1,$$
$$U' = -\frac{2V}{\pi}\log\frac{4}{k} + 1,$$
$$V' = -\frac{2U}{\pi}\log\frac{4}{k} + 1,$$
$$W' = -\frac{2W}{\pi}\log\frac{4}{k} + 1.$$

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§ 14. Substituting for K, E, &c. and for K_2, E_2, &c. their approximate values given in § 12, we have

$$K' = (1+\tfrac{1}{4}k^2)\log\frac{4}{k} - \tfrac{1}{4}k^2,$$

$$E' = \tfrac{1}{2}k^2\log\frac{4}{k} + 1 - \tfrac{1}{4}k^2,$$

$$I' = -(1-\tfrac{1}{4}k^2)\log\frac{4}{k} + 1 - \tfrac{5}{128}k^4,$$

$$G' = -\tfrac{1}{2}k^2\log\frac{4}{k} + 1 - \tfrac{1}{4}k^2,$$

$$U' = -\tfrac{1}{2}(1+\tfrac{1}{4}k^2)\log\frac{4}{k} + 1 - \tfrac{1}{8}k^2,$$

$$V' = \tfrac{1}{16}k^4\log\frac{4}{k} + 1 - \tfrac{1}{4}k^2,$$

$$W' = -\tfrac{1}{2}(1-\tfrac{3}{4}k^2)\log\frac{4}{k} + 1 - \tfrac{1}{8}k^2.$$

Writing the terms in order of magnitude and retaining them as far as those involving k^2, we obtain the formulæ:

$$K' = -\log k + 2\log 2 - \tfrac{1}{4}k^2\log k + \tfrac{1}{4}k^2(2\log 2 - 1),$$

$$E' = 1 - \tfrac{1}{2}k^2\log k + \tfrac{1}{4}k^2(4\log 2 - 1),$$

$$I' = \log k + 1 - 2\log 2 - \tfrac{1}{4}k^2\log k + \tfrac{1}{2}k^2\log 2,$$

$$G' = 1 + \tfrac{1}{2}k^2\log k - \tfrac{1}{4}k^2(4\log 2 + 1),$$

$$U' = \tfrac{1}{2}\log k + 1 - \log 2 + \tfrac{1}{8}k^2\log k - \tfrac{1}{8}k^2(2\log 2 + 1),$$

$$V' = 1 - \tfrac{1}{4}k^2,$$

$$W' = \tfrac{1}{2}\log k + 1 - \log 2 - \tfrac{3}{8}k^2\log k + \tfrac{1}{8}k^2(6\log 2 - 1).$$

§ 15. Neglecting terms of the order $k^2\log k$,

$$K' = \log\frac{4}{k},$$

$$E' = 1,$$

$$I' = 1 - \log\frac{4}{k},$$

$$G' = 1,$$

$$U' = 1 - \tfrac{1}{2}\log\frac{4}{k},$$

$$V' = 1,$$

$$W' = 1 - \tfrac{1}{2}\log\frac{4}{k}.$$

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Differential equations satisfied by K and K', E and I', &c., §§ 16—18.

§ 16. It is well known that K and K' satisfy the same differential equation of the second order, viz.

$$k(1-k^2)\frac{d^2u}{dk^2}+(1-3k^2)\frac{du}{dk}-ku=0,$$

and it was in fact by means of this differential equation that Legendre obtained the series for K' in terms of k. It is well known also that the quantities E and $K'-E'$ satisfy the same differential equation; whence also E' and $K-E$ satisfy the same differential equation. It can be shewn that the same differential equations are satisfied by G and G', by U and V', by V and U', and by W and W'. These seven differential equations and their complete integrals are as follows:

(i)

$$k(1-k^2)\frac{d^2u}{dk^2}+(1-3k^2)\frac{du}{dk}-ku=0,$$

$$u=c_1K+c_2K';$$

(ii)

$$k(1-k^2)\frac{d^2u}{dk^2}+(1-k^2)\frac{du}{dk}+ku=0,$$

$$u=c_1E+c_2I';$$

(iii)

$$k(1-k^2)\frac{d^2u}{dk^2}-(1+k^2)\frac{du}{dk}+ku=0,$$

$$u=c_1I+c_2E';$$

(iv)

$$k(1-k^2)\frac{d^2u}{dk^2}-(1-k^2)\frac{du}{dk}-ku=0,$$

$$u=c_1G+c_2G';$$

(v)

$$k(1-k^2)\frac{d^2u}{dk^2}-(3-k^2)\frac{du}{dk}-ku=0,$$

$$u=c_1U+c_2V';$$

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(vi)

$$k(1-k^2)\frac{d^2u}{dk^2}+(1+k^2)\frac{du}{dk}-ku=0,$$

$$u=c_1V+c_2U';$$

(vii)

$$k(1-k^2)\frac{d^2u}{dk^2}+(1-3k^2)\frac{du}{dk}+3ku=0,$$

$$u=c_1W+c_2W',$$

where c_1 and c_2 are arbitrary constants.

§ 17. These differential equations assume a more elegant form when the independent variable is taken to be k^2 instead of k.

Putting $h=k^2, \quad h'=k'^2,$

so that $h+h'=1,$

the differential equations are:

(i)

$$4hh'\frac{d^2u}{dh^2}+4(h'-h)\frac{du}{dh}-u=0,$$

$$u=c_1K+c_2K';$$

(ii)

$$4hh'\frac{d^2u}{dh^2}+4h'\frac{du}{dh}+u=0,$$

$$u=c_1E+c_2I';$$

(iii)

$$4hh'\frac{d^2u}{dh^2}-4h\frac{du}{dh}+u=0,$$

$$u=c_1I+c_2E';$$

(iv)

$$4hh'\frac{d^2u}{dh^2}\qquad\qquad-u=0.$$

$$u=c_1G+c_2G';$$

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(v)

$$4hh'\frac{d^2u}{dh^2}-4\frac{du}{dh}-u=0,$$

$$u=c_1U+c_2V';$$

(vi)

$$4hh'\frac{d^2u}{dh^2}+4\frac{du}{dh}-u=0,$$

$$u=c_1V+c_2U';$$

(vii)

$$4hh'\frac{d^2u}{dh^2}+4(h'-h)\frac{du}{dh}+3u=0,$$

$$u=c_1W+c_2W'.$$

§ 18. The differential equations may be very easily obtained in the form given in the last section by means of the differential formulæ:

$$K=2\frac{dG}{dh}, \qquad G=2hh'\frac{dK}{dh},$$

$$E=-2h'\frac{dI}{dh}, \qquad I=2h\frac{dE}{dh},$$

which give

$$\frac{dU}{dh}=-\frac{G}{4h'}, \qquad \frac{dV}{dh}=\frac{G}{4h},$$

$$\frac{dW}{dh}=\frac{h'I-hE}{4hh'}.$$

Definitions of K, E, I, &c. as integrals, §§ 19, 20.

§ 19. Denoting by X the expression

$$\sqrt{\{(1-x^2)(1-k^2x^2)\}},$$

the definitions of K, E, I, &c. as integrals are

(i) $$K=\int_0^1\frac{1}{X}dx,$$

(ii) $$E=\int_0^1\frac{1-k^2x^2}{X}dx,$$

(iii) $$I=-\int_0^1\frac{k^2x^2}{X}dx,$$

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$$\text{(iv)} \qquad G = \int_0^1 \frac{k^2 - k^2x^2}{X}\, dx,$$

$$\text{(v)} \qquad U = \int_0^1 \frac{\frac{1}{2}k^2 - k^2x^2}{X}\, dx,$$

$$\text{(vi)} \qquad V = \int_0^1 \frac{\frac{1}{2}(1+k^2) - k^2x^2}{X}\, dx,$$

$$\text{(vii)} \qquad W = \int_0^1 \frac{\frac{1}{2} - k^2x^2}{X}\, dx,$$

the track of integration in each case being real.

§ 20. If the track be not real, but be the same for all the integrals, the corresponding simultaneous values of the integrals are

$$\text{(i)} \qquad (2m+1)K + 2niK',$$

$$\text{(ii)} \qquad (2m+1)E - 2niI',$$

$$\text{(iii)} \qquad (2m+1)I - 2niE',$$

$$\text{(iv)} \qquad (2m+1)G - 2niG',$$

$$\text{(v)} \qquad (2m+1)U - 2niV',$$

$$\text{(vi)} \qquad (2m+1)V - 2niU',$$

$$\text{(vii)} \qquad (2m+1)W - 2niW'.$$

Note on the Differential Equations, § 21.

§ 21. Defining K, E, I, &c. as in § 19 by means of the integrals, we may shew, by differentiation and transformation of these integrals, that they satisfy the differential equations (i), ... (vii). The validity of the processes is in no way dependent upon the reality of the track of integration, though it must be the same in the case of each of the integrals used in verifying the same differential equation. The seven differential equations are therefore satisfied respectively by the seven values of the integrals given in the last section. Thus the differential equation satisfied by K is satisfied by K' also, that satisfied by E is satisfied by I' also, and so on. This explanation of the fact that K and K', E and I', &c. satisfy the same differential equations is well known: it is only referred to here in order to notice that the same explanation applies in the case of each of the seven differential equations.

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Solution of differential equations by series, § 22.

§ 22. In the solution of differential equations by series it is well known that, supposing the equation to be linear and of the second order, if we obtain as particular integrals two series R and S, then the general integral of the equation is

$$u = c_1 R + c_2 S;$$

but that if in the formation of the series S the coefficient of a term becomes infinite, the general integral is of the form

$$u = aR \log bx + T,$$

T being a new series, and a and b being the arbitrary constants. The seven differential equations afford examples of this principle. Taking, for example, the fourth equation of § 17, viz.

$$4h(1-h)\frac{d^2u}{dh^2} - u = 0,$$

and, following the usual process, let

$$u = \Sigma A_r h^{m+r},$$

the summation extending to all positive integral values of r; then, substituting in the differential equation, we have

$$4(m+r)(m+r-1)A_r - (2m+2r-3)^2 A_{r-1} = 0,$$

whence, putting $r = 0$, we find

$$m = 0 \text{ or } 1.$$

The equations giving $A_1, A_2, A_3 \ldots$ are

$$2m(2m+2)A_1 - (2m-1)^2 A_0 = 0,$$

$$(2m+2)(2m+4)A_2 - (2m+1)^2 A_1 = 0,$$

$$(2m+4)(2m+6)A_3 - (2m+3)^2 A_2 = 0,$$

&c. &c.

Taking the root $m = 1$,

$$A_1 = \frac{1^2}{2.4} A_0,$$

$$A_2 = \frac{3^2}{4.6} A_1,$$

$$A_3 = \frac{5^2}{6.8} A_2,$$

&c.

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We thus obtain as a particular integral

$$u = A_0 \left(h + \frac{1^2}{2.4} h^2 + \frac{1^2.3^2}{2.4^2.6} h^4 + \frac{1^2.3^2.5^2}{2.4^2.6^2.8} h^6 + \&c. \right),$$

that is, in the notation of § 9,

$$u = A_0 . 2G_1.$$

If we take the root $m = 0$ we obtain infinite values for A_1, A_2, &c. and we conclude therefore that the general integral of the equation is of the form

$$u = aG_1 \log bh + T,$$

the coefficients in the series T being determined by substituting this expression for u in the differential equation and equating coefficients.

Since G' satisfies the differential equation it must be of this form, and comparing its value, viz.

$$\tfrac{1}{2} G_1 \log \tfrac{1}{16} h + G_2,$$

with

$$aG_1 \log bh + T,$$

we see that it is included as the particular case $a = \frac{1}{2}$, $b = \frac{1}{16}$, the series T being equal to G_2.

In the solution of each of the seven equations we obtain directly by the ordinary process of integration one series proceeding by ascending powers of h. These are the series K_1, E_1, I_1, &c. The series K_2, E_2, I_2, &c. are the values of T in the different cases.

Series for K, G, W, involving sines of multiples of the modular angle, § 23.

§ 23. In Vol. XIX. (pp. 51, 52) of *Crelle's Journal,* Gudermann has given the following remarkable formula in which θ denotes the modular angle:

$$K = \pi \left(\sin\theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2.3^2}{2^2.4^2} \sin 9\theta + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \sin 13\theta + \&c. \right).$$

The chief interest of this formula consists in its elegance, as it is of course not so suitable for the calculation of K as the series proceeding by powers of k^2. The method by which Gudermann obtained the above series for K is in effect as follows:

By the change of k' into $\dfrac{1-k}{1+k}$, K' is changed into $\frac{1}{2}(1+k)K'$,

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and by the change of k into $\frac{ik}{k'}$, K' is changed into $k'(K'-iK)$. Thus, by the double transformation,

$$k' \text{ is changed into } \frac{k'-ik}{k'+ik} = e^{-2i\theta},$$

and K' „ „ $\frac{1}{2}(k'+ik)(K'-iK) = \frac{1}{2}e^{i\theta}(K'-iK)$.

Replacing k by k' in the first formula in § 2, we have

$$\frac{2K'}{\pi} = 1 + \frac{1^2}{2^2}k'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k'^6 + \&c.,$$

whence, changing k' into $e^{-2i\theta}$,

$$\frac{e^{i\theta}(K'-iK)}{\pi} = 1 + \frac{1^2}{2^2}e^{-4i\theta} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}e^{-8i\theta} + \&c.,$$

that is,

$$\frac{K'-iK}{\pi} = e^{-i\theta} + \frac{1^2}{2^2}e^{-5i\theta} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}e^{-9i\theta} + \&c.$$

$$= \cos\theta + \frac{1^2}{2^2}\cos 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\cos 9\theta + \&c.$$

$$- i\left(\sin\theta + \frac{1^2}{2^2}\sin 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin 9\theta + \&c.\right),$$

giving

$$\frac{K'}{\pi} = \cos\theta + \frac{1^2}{2^2}\cos 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\cos 9\theta + \&c.,$$

and

$$\frac{K}{\pi} = \sin\theta + \frac{1^2}{2^2}\sin 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin 9\theta + \&c.$$

We may obtain corresponding formulæ for G and W by a similar method, as follows.

§ 24. The change of k' into $\frac{1-k}{1+k}$ corresponds to the change of q into $q^{\frac{1}{2}}$, and the change of k into $\frac{ik'}{k}$ corresponds to the change of q into $-q$. The double transformation therefore corresponds to the change of q into $iq^{\frac{1}{2}}$.

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By the change of q into $q^{\frac{1}{2}}$,

$$k' \text{ becomes } \frac{1-k}{1+k},$$

$$K' \quad \text{,,} \quad \tfrac{1}{2}(1+k)K',$$

$$I' \quad \text{,,} \quad \frac{U'}{1+k},$$

$$V' \quad \text{,,} \quad \frac{E'}{1+k};$$

and, by the change of q into $-q$,

$$k \text{ becomes } \frac{ik}{k'},$$

$$K' \quad \text{,,} \quad k'(K'-iK),$$

$$U' \quad \text{,,} \quad \frac{W'+iW}{k'},$$

$$E' \quad \text{,,} \quad \frac{G'+iG}{k'}.$$

Thus, by the change of q into $iq^{\frac{1}{2}}$,

$$k' \text{ becomes } e^{-2i\theta},$$

$$K' \quad \text{,,} \quad \tfrac{1}{2}e^{i\theta}(K'-iK),$$

$$I' \quad \text{,,} \quad e^{-i\theta}(W'+iW),$$

$$V' \quad \text{,,} \quad e^{-i\theta}(G'+iG).$$

§ 25. Starting with the formulæ,

$$\frac{2K'}{\pi} = 1 + \frac{1^2}{2^2}k'^2 + \frac{1^2.3^2}{2^2.4^2}k'^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}k'^6 + \&c.,$$

$$\frac{2I'}{\pi} = -\frac{1}{2}k'^2 - \frac{1^2.3}{2^2.4}k'^4 - \frac{1^2.3^2.5}{2^2.4^2.6}k'^6 - \&c.,$$

$$\frac{4V'}{\pi} = 1 + \frac{1}{2^2}k'^2 + \frac{1^2}{2^2.4^2}k'^4 + \frac{1^2.3^2}{2^2.4^2.6^2}k'^6 + \&c.,$$

we find, by replacing k' by $e^{-2i\theta}$, and equating the real and imaginary parts of the expressions as in § 23,

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$$\frac{K}{\pi} = \sin\theta + \frac{1^2}{2^2}\sin 5\theta + \frac{1^2.3^2}{2^2.4^2}\sin 9\theta + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\sin 13\theta + \&c.,$$

$$\frac{K'}{\pi} = \cos\theta + \frac{1^2}{2^2}\cos 5\theta + \frac{1^2.3^2}{2^2.4^2}\cos 9\theta + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cos 13\theta + \&c.,$$

$$\frac{2W}{\pi} = \frac{1}{2}\sin 3\theta + \frac{1^2.3}{2^2.4}\sin 7\theta + \frac{1^2.3^2.5}{2^2.4^2.6}\sin 11\theta + \&c.,$$

$$\frac{2W'}{\pi} = -\frac{1}{2}\cos 3\theta - \frac{1^2.3}{2^2.4}\cos 7\theta - \frac{1^2.3^2.5}{2^2.4^2.6}\cos 11\theta - \&c.,$$

$$\frac{4G}{\pi} = \sin\theta - \frac{1}{2^2}\sin 3\theta - \frac{1^2}{2^2.4^2}\sin 7\theta - \frac{1^2.3^2}{2^2.4^2.6^2}\sin 11\theta - \&c.,$$

$$\frac{4G'}{\pi} = \cos\theta + \frac{1^2}{2^2}\cos 3\theta + \frac{1^2}{2^2.4^2}\cos 7\theta + \frac{1^2.3^2}{2^2.4^2.6^2}\cos 11\theta + \&c.$$

The series for K and W assume the form $0 \times \infty$ when $\theta = 0$. When $\theta = \frac{1}{2}\pi$, they are infinite in value, as they should be. Except in these critical cases, and the corresponding cases for K' and W', the series are convergent for all values of θ.

Similar series for E and I, § 26.

§ 26. Since

$$E = W + \tfrac{1}{2}K,$$

and

$$I = W - \tfrac{1}{2}K,$$

we may at once deduce from the series for $\frac{K}{\pi}$ and $\frac{2W}{\pi}$ in the last section, the following series for $\frac{2E}{\pi}$ and $\frac{2I}{\pi}$:

$$\frac{2E}{\pi} = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1^2}{2^2}\sin 5\theta + \frac{1^2.3}{2^2.4}\sin 7\theta + \frac{1^2.3^2}{2^2.4^2}\sin 9\theta + \frac{1^2.3^2.5}{2^2.4^2.6}\sin 11\theta + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\sin 13\theta + \&c.,$$

$$\frac{2I}{\pi} = -\sin\theta + \frac{1}{2}\sin 3\theta - \frac{1^2}{2^2}\sin 5\theta + \frac{1^2.3}{2^2.4}\sin 7\theta - \frac{1^2.3^2}{2^2.4^2}\sin 9\theta + \frac{1^2.3^2.5}{2^2.4^2.6}\sin 11\theta - \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\sin 13\theta + \&c.$$

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Legendre's formulæ for K' and E', §§ 27, 28.

§ 27. The series for K' and E' in terms of k are due to Legendre. Changing only his b, F^1, E^1 into k, K', E', the following is the form in which they were given by him*:

$$K' = \log\frac{4}{k} + \frac{1^2}{2^2}k^2\left(\log\frac{4}{k} - 1\right) + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4\left(\log\frac{4}{k} - 1 - \frac{2}{3 \cdot 4}\right) + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6\left(\log\frac{4}{k} - 1 - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) + \&c.$$

$$E' = 1 + \frac{1}{2}k^2\left(\log\frac{4}{k} - \frac{1}{1 \cdot 2}\right) + \frac{1^2}{2^2} \cdot \frac{3}{4}k^4\left(\log\frac{4}{k} - A' - \frac{1}{3 \cdot 4}\right) + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{5}{6}k^6\left(\log\frac{4}{k} - A'' - \frac{1}{5 \cdot 6}\right) + \&c.,$$

where

$$A' = 1,$$

$$A'' = 1 + \frac{2}{3 \cdot 4},$$

$$A''' = 1 + \frac{2}{3 \cdot 4} + \frac{2}{5 \cdot 6},$$

&c. &c.

§ 28. Legendre remarks that the coefficients in the series for E' agree with those which he had given under another form in the *Mémoires de l'Académie* for 1786.

* *Traité des Fonctions Elliptiques*, Vol. I. pp. 67, 68.

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Changing Legendre's notation as in the last section, the formula in the *Mémoires** becomes

$$E' = 1$$
$$+\left(k^2+\frac{3}{4}\frac{1}{2}k^4+\frac{3.5}{4.6}\frac{1.3}{2.4}k^6+\frac{3.5.7}{4.6.8}\frac{1.3.5}{2.4.6}k^8+\&c.\right)\log\frac{2}{\sqrt{k}}$$
$$-k^2\frac{1}{4}$$
$$-\frac{3}{4}\cdot\frac{1}{2}k^4\left(\frac{7}{12}-\frac{1}{2.3.4}\right)$$
$$-\frac{3.5}{4.6}\cdot\frac{1.3}{2.4}k^6\left(\frac{13}{20}-\frac{1}{2.3.4}-\frac{1}{4.5.6}\right)$$
$$-\frac{3.5.7}{4.6.8}\cdot\frac{1.3.5}{2.4.6}k^8\left(\frac{19}{28}-\frac{1}{2.3.4}-\frac{1}{4.5.6}-\frac{1}{6.7.8}\right)$$
$$-\&c.$$

In this series the general term is

$$\frac{3.5\ldots(2n-1)}{4.6\ldots 2n}\cdot\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}k^{2n}S_n,$$

where

$$S_n=\frac{6n-5}{4(2n-1)}-\frac{1}{2.3.4}-\frac{1}{4.5.6}\cdots-\frac{1}{(2n-2)(2n-1)2n}.$$

To identify this term with the corresponding term in the series for E' in § 10, we notice that

$$\frac{6n-5}{2(2n-1)}=2-\frac{1}{2}-\frac{1}{2n-1},$$

and

$$\frac{2}{r(r+1)(r+2)}=\frac{1}{r}-\frac{2}{r+1}+\frac{1}{r+2}.$$

Thus $2S_n=$

$$2-\frac{1}{2}-\frac{1}{2n-1}-\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)-\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right)\cdots-\left(\frac{1}{2n-2}-\frac{2}{2n-1}+\frac{1}{2n}\right)$$
$$=2-\frac{2}{2}+\frac{2}{3}\cdots-\frac{2}{2n-2}+\frac{1}{2n-1}-\frac{1}{2n},$$

and the coefficient of k^{2n} is evidently the same as in I_2 § (9).

* *Mémoire sur les Intégrations par arcs d'ellipse*, p. 630.

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Gudermann's form of the series for K' and E', § 29.

§ 29. In Vol. XIX. pp. 55—58 of *Crelle's Journal*, Gudermann gave the series for K' and E' in Legendre's form and also in the following slightly different form:

Let

$$\mathfrak{K} = 1 + \frac{1^2}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6 + \&c.$$

and

$$\mathfrak{K}_n = 1 + \frac{1^2}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 \ldots + \frac{1^2 \cdot 3^2 \cdot 5 \ldots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \ldots (2n)^2}k^{2n},$$

(so that $\mathfrak{K}_n$ denotes the first $n+1$ terms of $\mathfrak{K}$), then

$$K' = \mathfrak{K} \log \frac{4}{k} - (\mathfrak{K} - 1) - \frac{2}{3 \cdot 4}(\mathfrak{K} - \mathfrak{K}_1) - \frac{2}{5 \cdot 6}(\mathfrak{K} - \mathfrak{K}_2) - \&c.$$

Similarly, let

$$t = \frac{1}{2}k^2 + \frac{1^2 \cdot 3}{2^2 \cdot 4}k^4 + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6}k^6 + \&c.,$$

and

$$t_n = \frac{1}{2}k^2 + \frac{1^2 \cdot 3}{2^2 \cdot 4}k^4 \ldots + \tfrac{1}{2} \cdot \frac{1^2 \cdot 3^2 \ldots (2n-3)^2 (2n-1)}{2^2 \cdot 4^2 \ldots (2n-2)^2 \, 2n}k^{2n},$$

(so that t_n denotes the first n terms of t, the last term being halved), then

$$E' = t \log \frac{4}{k} + 1 - (t - t_1) - \frac{2}{3 \cdot 4}(t - t_2) - \frac{2}{5 \cdot 6}(t - t_3) - \&c.$$

It does not seem worth while to give the corresponding forms of the series for I', G', &c.; they may be derived at once from the formulæ in § 3.

Weierstrass's J and J', § 30.

§ 30. In Weierstrass's notation, in which $K - E$ is denoted by J and E' by J' (see § 7), we have, as noticed by Weierstrass himself*, the corresponding approximate formulæ:

$$K' = \frac{2K}{\pi} \log \frac{4}{k},$$

$$J' = 1 + \frac{2J}{\pi} \log \frac{4}{k}.$$

* *Crelle's Journal*, Vol. LII. p. 364.

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This second equation in the notation of this paper is

$$E' = -\frac{2I}{\pi} \log \frac{4}{k} + 1,$$

and is the second of the system of six formulæ in § 13.

(2) *On a simplified form of Apparatus for determining the density of Ozone.* By C. T. HEYCOCK, M.A., and F. H. NEVILLE, M.A.

IN 1868, M. Soret published in the *Ann. Chim. Phys.* t. 157 his researches on the density of ozone based upon its rate of diffusion.

The corrosive action and the instability of ozone led him to devise a very complicated apparatus. The following is an account of a much simplified form, for the same purpose, as used by the authors.

Their apparatus was made of two gas cylinders of thick glass each containing about 300 cc., the mouths of the cylinders being carefully ground. The closed ends of the cylinders were pierced with a hole and each was fitted by an accurately ground tube and stopcock.

Care must be taken to grind the tubes to fit accurately the holes in the cylinders as all lutes are unadvisable in the presence of ozone. The mouth of each jar is closed by a thick sheet of plate glass, which we call dampers, rather wider than the jar and about twice this in length. Each damper was perforated by a round hole, the diameter of the smaller hole being about 4 mm. and the other about 10 mm.

The experiments were performed as follows:—one jar was clamped with its mouth upwards, and on it were laid the two dampers, and upon them the other jar mouth downwards. By sliding the dampers it is easy to bring the jars into communication either with each other or with the air. The jars having been brought into communication and the stopcocks opened oxygen gas was blown through until the air was displaced.

The taps were then shut and communication between the jars interrupted by sliding the dampers. The lower jar by the same means was brought into communication with the air and an approximately known quantity, either chlorine or ozone, blown into it. The lower jar was then closed and the apparatus left to rest for some hours to get rid of convection and other currents. The taps were then opened for a moment to equalize the pressure and

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the jars brought into communication by sliding the dampers until the holes were concentric.

The diffusion thus established was allowed in all our experiments to proceed for 45 minutes, after which the dampers were closed.

The quantity of chlorine or ozone present in *each* jar was then determined by absorption with KI, and subsequent titration by a solution of $Na_2S_2O_3$. Whatever be the chemical reactions, the amount of Iodine liberated, and therefore the number of cc. of $Na_2S_2O_3$ used is in every case proportional to the amount of chlorine or ozone present in the upper and lower jar. (For convenience, we used a standard solution of $Na_2S_2O_3$ to obtain an idea of the quantity of gas used in each experiment.)

Let v and V be the number of cc.'s of $Na_2S_2O_3$ required for the upper and lower jars respectively when chlorine was the gas diffused, then $\frac{v}{V+v}$ = ratio of amount of chlorine diffused to the whole amount of chlorine taken.

This ratio is independent of the quantity of diffusing gas originally placed in the lower jar but is a function of the time, the temperature, the size of the smaller hole, and the density of the gases. If the time of diffusion were long enough this ratio would become $\frac{1}{2}$, but when the time is comparatively short the ratio $\frac{v}{V+v}$ may be taken to be proportional to the velocity of diffusion of the particular gas, that is to $\frac{1}{\sqrt{D}}$ where D = density.

The mean of six determinations with chlorine gave

$$\frac{v}{V+v} = 0{\cdot}117.$$

The mean of fourteen determinations with ozone gave

$$\frac{v}{V+v} = 0{\cdot}147.$$

Hence $$\frac{(117)^2}{(147)^2} = \frac{\text{density of ozone}}{\text{density of chlorine}}.$$

Assuming the density of chlorine to be 35·5 we get from this equation the density of ozone to be 22·5, a result which sufficiently justifies the formula O_3 for the molecule of ozone.

The following table gives the actual numbers recorded in our experiments.

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CHLORINE.

Top Jar v. Chlorine diffused in 45 min. estimated in cc.'s of $Na_2S_2O_3$ required for liberated Iodine.	Total $V+v$. Chlorine in lower jar at commencement of exp. Stated in cc.'s of $Na_2S_2O_3$.	Ratio $\frac{v}{V+v}$.
1·75	15·6	0·112
5·6	48·7	0·115
5·85	49·95	0·117
13·88	118·6	0·117
17·1	144·7	0·118
27·45	228·6	0·120
	Mean	0·117

1 cc. of $Na_2S_2O_3 = 0·208$. cc. of Cl at 0°C. and 760mm.

Hence the largest quantity of Cl diffused in one experiment was 5·71 cc.

OZONE.

Top Jar v. Diffused Ozone stated in cc. of $Na_2S_2O_3$ required for liberated Iodine.	Total $V+v$. Ozone in lower jar at commencement stated in cc.'s of $Na_2S_2O_3$.	Ratio $\frac{v}{v+V}$.
0·95	6·45	0·1473
1·43	9·93	0·1436
2·03	15·00	0·1350
2·27	15·75	0·1444
2·73	18·85	0·1446
3·28	22·08	0·1483
3·50	22·9	0·1530
4·83	32·05	0·1505
6·43	42·03	0·1530
6·60	42·93	0·1538
6·7	43·7	0·1533
8·05	53·60	0·1498
8·38	54·05	0·1514
9·63	63·68	0·1510
	Mean	0·1470

1 cc. $Na_2S_2O_3 = 0·208$. cc. of Ozone at 0°C. and 760 mm.

if we accept the usual reaction for this gas.

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Hence largest quantity of ozone diffused was about 2 cc.

We hope shortly to apply this method to other gases and vapours; the chief difficulty however will be to find methods sufficiently delicate for the estimation of small quantities.

If the gas experimented upon, and the gas into which it is diffused be of very different densities, a considerable source of error will be introduced through *mass motion.*

If we follow the motion of the chlorine molecules in the lower jar and of an equal number of oxygen molecules in the upper jar, we see that when diffusion has commenced the more rapidly moving oxygen molecules will enter the lower jar by diffusion in greater number than the chlorine molecules can enter the upper jar. There will therefore be a tendency for molecules to accumulate in the lower jar, in other words for the pressure to rise, as is the case when a porous septum is used, and therefore there will be a continuous efflux of mixed gases from the lower jar not due to diffusion proper. This will tend to make v and therefore the ratio too great.

Trial proved that the apparatus was gas tight for considerable differences of pressure; however, to gain additional security we frequently lubricated our dampers with concentrated H_2SO_4.

The absorption of the chlorine at the end of the diffusion was effected by placing each jar closed by its damper mouth downwards in a solution of KI, and the solution allowed to come in contact with the gas.

The oxygen in the jar was then finally washed from the last traces of chlorine by blowing up a fine jet of KI solution from a curved pipette. The jars containing the ozone were treated in the same way. When the ozone came in contact with the KI the jars became filled with a dense white nearly opaque fog, the *autozone* of Schönbein.

In these experiments the time allowed for each diffusion was that adopted by Soret, and the size of the diffusion hole was rather less than his.

We think it will be worth while to enlarge the diffusion hole considerably so as to be able to shorten the time.

(3) *On the effects of self-induction of the galvanometer in the determination of the capacity of a condenser.* By J. C. M'Connel, B.A.

In the calculation which Mr Thomson gives with reference to the method of determining the capacity of a condenser, to which your attention has been so lately drawn by Mr Wilberforce, there is a point of some importance which he passes by without notice.

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He neglects the self-induction of all the resistances. Now one of these resistances is the galvanometer, and the coefficient of self-induction of a galvanometer of 11,000 ohms resistance is by no means small. My first impression was that the total quantity of electricity, that passed through the galvanometer in the transient current which charges the condenser, would be considerably diminished by the self-induction. Although this proved to be not the case, the results I obtained seemed to me to be of sufficient interest to be worthy of your notice.

To prevent the physical peculiarities of the motion from being obscured by the length of the algebra, let us first consider a simple case which has very similar characteristics.

A charged condenser is permitted to discharge itself through two resistances placed in parallel arc, only one of which has appreciable self-induction.

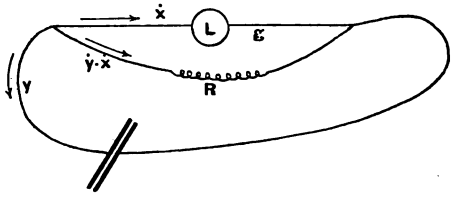
Let g be the resistance which has self-induction L,

R the other resistance,

$\dot{x}$ the current through g,

$\dot{y}$ the current from the condenser of capacity C.

E the potential to which the condenser is initially charged.



We have the equations

$$\left.\begin{aligned} L\ddot{x} + g\dot{x} - R(\dot{y} - \dot{x}) &= 0 \\ (\dot{y} - \dot{x})\, R &= E - \frac{y}{C} \end{aligned}\right\},$$

or

$$\left.\begin{aligned} L\ddot{x} + (g+R)\,\dot{x} - R\dot{y} &= 0 \\ R\dot{x} - R\dot{y} - \frac{y}{C} + E &= 0 \end{aligned}\right\} \begin{aligned} &\ldots\ldots\ldots\ldots\ldots (1), \\ &\ldots\ldots\ldots\ldots\ldots (2). \end{aligned}$$

At first sight I thought that since the self-induction delayed the current through the branch g, the greater part of the discharge would pass through the branch R and thus the whole current through g would be diminished.

But the investigation below shews that though the self-induction prevents the current from attaining its full magnitude at

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once, and diminishes the magnitude to which it does attain, it is yet so effective in maintaining the current that the deficiency is exactly neutralised.

It appears further that in certain cases the self-induction instead of delaying the transient currents actually accelerates their disappearance.

Integrating equation (1) we get

$$L\dot{x} + (g + R)\, x - Ry = \text{constant}.$$

Now when $t = 0, \quad \dot{x} = 0, \quad y = 0,$

$$t = \infty, \quad \dot{x} = 0, \quad y = CE.$$

Thus when $t = \infty$, $x = \dfrac{R}{g + R} CE$ and is independent of L.

Though L has no effect on the magnitude of the total currents, it has an important effect on their duration.

Eliminating y between our two equations, we get

$$CRL\ddot{x} + (L + CRg)\, \dot{x} + (g + R)\, x = CER.$$

The solution takes different forms according as the roots of the equation

$$CRL\xi^2 + (L + CRg)\, \xi + (g + R) = 0 \ldots\ldots\ldots\ldots (3)$$

are real or imaginary.

When the roots are real

$$\dot{x} = \frac{E}{2\mu L} e^{-\lambda t} (e^{\mu t} - e^{-\mu t}),$$

where $\lambda = \dfrac{L + CRg}{2CRL}$, $\mu^2 = \dfrac{(L + CRg)^2 - 4CRL\,(g + R)}{4C^2R^2L^2}$.

When the roots are imaginary

$$\dot{x} = \frac{E}{\nu L} e^{-\lambda t} \sin \nu t,$$

where $\nu^2 = \dfrac{4CRL\,(g + R) - (L + CRg)^2}{4C^2R^2L^2}$.

Thus when L is small $\dot{x}$ never changes sign. It rises to a certain value and then falls again to zero.

When L is increased till it is greater than a certain value α_1, the motion becomes oscillatory, the condenser is discharged and charged again with a less charge of electricity of opposite sign, discharged again and so on.

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When L is increased still further till it exceeds a certain value α_2, the motion ceases to be oscillatory and assumes its former character.

Here α_1 and α_2 are the roots of the equation

$$\alpha^2 - 2CR\alpha\,(g + 2R) + C^2R^2g^2 = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

and are always real.

The practical question is usually to decide how soon the transient currents will die away. Let us then consider how L affects the magnitude of the time constants.

When L is very small

$\lambda + \mu$ is very large and is equal to $\frac{g}{L}$ approximately,

$\lambda - \mu$ is equal to $\frac{g+R}{CRg}$ approximately.

The larger depends mainly on the self-induction, the smaller mainly on the capacity.

As L increases, $\lambda - \mu$ increases till $L = \alpha_1$.

Thus a moderate amount of self-induction has the effect of diminishing the duration of the transient currents.

While the motion is oscillatory λ diminishes as L increases.

When $L > \alpha_2$, $\lambda - \mu$ diminishes as L increases and continually approximates to the value $\frac{g+R}{L}$, $\lambda + \mu$ continually approximates to the value $\frac{1}{CR}$. Thus the smaller time constant is governed by the self-induction, the larger by the capacity.

When L is very large the behaviour of the currents is sufficiently curious to merit a fuller statement. The condenser first almost entirely discharges itself through the arm R. Then the current from the condenser becomes indefinitely small in comparison with the current through g, and the current in R changes sign. The current through g reaches its maximum value, which is small, before the condenser is nearly discharged, and when once started takes a long time to die away. Thus the only appreciable current, that survives after a short time, is the one flowing through g and back through R.

We may now pass to the more complicated arrangement which is actually employed in determining the capacity of a condenser.

The first figure is a sketch of the arrangement, of which Mr Wilberforce has given a full description.

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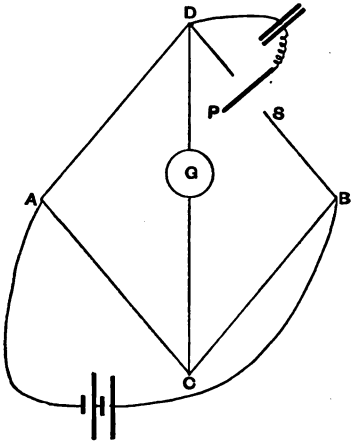
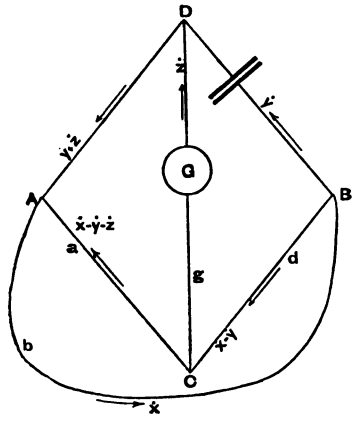
What we are at present considering is the whole current through the galvanometer due to the charging of the condenser.





Let us therefore make abstraction of the electromotive force of the battery and the whole system of steady currents due to it.

This system of currents entails a difference of potentials between B and D which we may call E_1.

When therefore the condenser is really discharged, it is on our present supposition charged to a potential $-E_1$; and when it is really charged, it is on our present supposition discharged.

Hence we have to investigate the whole current through G in the arrangement represented in the second figure when the condenser, which is initially charged to a potential $-E_1$, discharges itself.

If a be infinite the arrangement becomes identical with that already considered.

We have the equations

$$b\dot{x} + a(\dot{x} - \dot{y} - \dot{z}) + d(\dot{x} - \dot{y}) = 0 \quad \ldots\ldots\ldots\ldots\ldots (5),$$

$$-a(\dot{x} - \dot{y} - \dot{z}) + c(\dot{y} + \dot{z}) - d(\dot{x} - \dot{y}) + \frac{y}{C} = -E_1 \quad \ldots\ldots (6),$$

$$-a(\dot{x} - \dot{y} - \dot{z}) + c(\dot{y} + \dot{z}) + g\dot{z} + L\ddot{z} = 0 \quad \ldots\ldots\ldots (7).$$

Eliminating $\dot{x}$ between (5) and (7), we get an equation of the form

$$A\dot{y} + B\dot{z} + L\ddot{z} = 0,$$

where A and B are independent of L.

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Integrating $Ay + Bz + L\dot{z} = 0,$

for when $t = 0, \quad y = 0, \quad z = 0$ and $\dot{z} = 0.$

Now when $t = \infty, \quad y = -CE_1 \quad \dot{z} = 0,$

$$\therefore Bz = -ACE_1.$$

Thus the whole current through the galvanometer is independent of L. It is clear that this result depends on the following conditions:

(1) The self-induction is in one branch only.

(2) The currents in all the branches can be expressed as linear functions of the current in that branch, of the difference of potentials at its ends, and of certain other currents whose integral value is fixed.

Thus it would hold good if the galvanometer were linked with any system of conductors, and condensers charged initially to any potential, provided there were no electromotive forces in the system and no appreciable self-induction except in the galvanometer itself.

The equation for the time constants is the quadratic in ξ.

$$\begin{vmatrix} a+b+d & -(a+d) & -a \\ -(a+d) & a+c+d+\dfrac{1}{\xi C} & a+c \\ -a & a+c & a+c+g+\xi L \end{vmatrix} = 0,$$

or $\xi^2 CL\{bc+(b+c)(a+d)\} + \xi\{L(a+b+d)+C(b+c)(ad+ag+dg)$

$$+ Cbc(d+g)\} + \{(a+c+g)(b+d)+a(c+g)\} = 0 \ldots.(8).$$

What is practically required is to know that the smaller time constant is so large that it may be safely treated as infinite; so it is convenient to have an inferior limit in a simple form.

If we write equation (8) in the form $P\xi^2 + Q\xi + R = 0$, then if the roots be real it may be easily shewn that the smaller time constant is greater than R/Q; if the roots be imaginary the real time constant is $Q/2P$.

But it is not likely that much error will arise in practice through neglecting the self-induction in calculating the time constant. For when L is large enough to govern the smaller time constant, the latter approximates to the value $(g+R)L$, where R is a positive constant depending on the resistances. Now the self-induction of a galvanometer coil of given shape and size varies approximately as the square of the number of turns, as does also the resistance. Thus g/L will have much the same value in all similar galvanometers. A short time ago I determined the self-

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induction of a 'sensitive mirror galvanometer' experimentally. L was $5\frac{1}{2}$ earth quadrants, while g was 13,000 ohms. So g/L was about 2400 in this case. Even if the time of charging were only $\frac{1}{300}$th of a second, the error in the value of the capacity produced by treating the time constant as ∞ instead of 2,400, would be less than one part in a thousand.

(4) *On the future of naval warfare, with an exhibition and account of a submarine boat.* By H. MIDDLETON.

Abstract.

It is not proposed in my paper—of which the following is a very short abstract—to repeat the same comments and considerations, on the subject of the present condition of the British navy, with which the press has lately teemed.

However, I have treated briefly of those general principles of naval warfare which govern the design and construction of all ships of war whatever. Because, on the knowledge of those principles the utility of all inventions and innovations from established forms and models entirely depends.

In my paper I pointed out that in consequence of the application of steam to fleets, together with the fact that each ironclad can now carry so short store of coal that the operation of refilling its bunkers has to be repeated every four or five days, while large "bases" for the supply of the fleet's motive power have to be maintained, require the study of a set of conditions of a character similar to those which give birth to what is known as "strategy" when applied to the maintenance and conduct of armies on land.

While the large fleet of coal ships which must now-a-days always accompany ironclads operating on the "high seas" necessitate the invention of a kind of naval tactics which differs considerably from those applicable to the wooden ships with which England maintained her sovereignty of the sea in her wars with the great Napoleon.

Now a knowledge, and a sound knowledge, of these matters forms the only guide an inventor has in determining the channels of thought into which he should direct his ingenuity. And this knowledge teaches him the conditions of the problem he must strive to solve; however, the recapitulation of them must be left out of this abstract, and I but state briefly because of the difficulty—if not impossibility—of fulfilling them by ships which move over the surface of the sea. I propose to make them move under the surface, and have thus carried out the problem I proposed to myself for solution.

Three different kinds of submarine boats have been designed, which kinds I shall call Class (*a*), (*b*) and (*c*) respectively.

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Class (*a*). The boats of this form are intended for speed, they are of a shape which somewhat resembles a cigar and may be called "fish" boats. Moreover these boats are furnished with two lateral fins, two—or one—screw propellers, at the stern, and a dorsal and ventral fin for guiding them, &c. &c. These boats are intended to remain a more or less period under water; but even those capable of being submerged for a very considerable time are yet supposed to come to the surface and be used as super-marine boats when cruising. When submerged, the boats simply 'fly' in the surrounding fluid: just as the Guillemot can, and does fly in water, or a bird in the air. The boats of Class (*a*) are armed with guns capable of being fired against an object under water: or the guns can be fired when the boat is above water, as when attacking a fort on land. However, such form of attack would rarely be adopted as the boat has only to be submerged, when it can be quietly run past the shore batteries without their being able to fire a shot in the defence of the channel which they were intended to guard. All the boats of Class (*a*) are fitted with apparatus for removing submarine mines, torpedoes, &c. &c.; but such operations are better performed by the boats of Class (*b*), of which a few words shall be said later. When the boats of Class (*a*) are intended to attack fleets of ironclads (as at present constructed), a large part of their time is taken up in cruising, which they do with their funnels and a small portion of their hull [hardly more than the "conning tower"] above water. When ironclads are sighted and the boat is to be taken "into action," it is submerged some ten, or more, feet. And (its funnels having been previously withdrawn into the 'hull' of the vessel,) its submarine engines are set going.

Because there is no "exhaust" thrown outside of the boat, there is no increase of "back pressure" on the pistons owing to depth; and indeed the boat can be worked at any depth less than that at which its hull would be crushed by hydrostatic pressure. The time that the boat can remain under water is several hours: but the time that she can run at full speed under water is much less than the period she can remain submerged with her crew, if her engines are not working. In all the boats, however, the time of submersion when running at full speed is sufficient to enable them to get out of gunshot of a stationary foe. [And the chances of an enemy steaming exactly parallel to the line of the submarine boat's motion under water, so as to be along side of it when it is forced to come to the surface—I say the chances are not very great: so the ironclad, if it does move, is just as likely to move away from the submarine boat as towards it, this consideration of the time to get out of gunshot of a stationary foe has determined the construction, &c. &c. of the submarine engines.] Usually, on account of the extremely small surface which the boat exposes

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(even when above water in cruising), a distance of 2½ miles from an ironclad can easily be approached without the smallest danger. The boat then being submerged and a "full head" of steam "put on," the ironclad can be reached in some 15 minutes; and when within some few yards, a percussion shell is fired against the bottom of the utterly helpless and harmless ship. Now it is known that the most powerful pumps in the navy cannot keep the water down when it pours through a 20-inch (diameter) hole: hence the possibility of preventing a vessel from sinking which has an aperture of several square yards torn in her bottom by the explosion of a shell does not exist. Men, guns, stores, all—must sink.

* * * * * *

Such in brief, are the functions of the boat one kind of which was exhibited; and in leaving the subject I have only to add that the boat exhibited was not furnished with guns and armour, but both of which it was my design to have furnished it with (could I have managed to do so) before it was exhibited.

Class (*b*). Boats of this class are in like manner intended to transport men and the material of war beneath the sea. They are not intended for speed: but for carrying troops under water for short distances, as across the British Channel, which thus becomes one of the most dangerous frontiers to defend that a state can well possess. However, as no boats of this kind were exhibited, nothing more will be said of them here.

Class (*c*). As this class was only represented by a very poor model, half made [and badly constructed], I will only say that the motor here is electricity, and the boats are to be fitted with torpedoes and used in connection with fast cruisers.

With this I finish this brief abstract of my paper.

(5) *Note on a peculiar sense organ in Scutigera coleoptrata, one of the Myriapoda.* By F. G. HEATHCOTE, B.A., Trin. Coll., Cambridge.

The organ is situated on the ventral surface of the head at a short distance from the mouth, near the base of the mandibles. A slit-like opening in the ventral median line, between the maxillæ and the base of the mandibles, leads into a sac lined with chitin. This sac is of elongated pear-like form, its longest diameter being parallel to the ventral surface of the head. Two longitudinal folds in its dorsal roof project close together into the body of the sac, partially dividing it into two pouches with a deep narrow median recess between them. In the anterior region of the sac a lateral fold on each side of the sac projects parallel to the ventral surface of the head, thus forming a deep lateral recess on each side.

The effect of these lateral and median recesses is to form a free lip on the dorsal median and lateral ventral aspects of the pouches.

The surface of the sac, except at the folds and in the recesse

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formed by the folds, is closely lined by a number of chitinous hairs. Each hair consists of a short and thick elliptical basal portion, one end of which is prolonged into a long fine hair which projects into the sac while the other end terminates in a short stump-like point which projects internally through the chitinous lining of the sac. The chitin of the sac is not of uniform thickness. In the recesses and on the folds which bound them it is smooth, but in the pouches, where the hairs exist, it is thrown into a series of ridges and depressions which give it a reticulated appearance. The base of a hair projects into each depression.

Internally the hypodermic cells which form the ordinary matrix of the exoskeleton line the chitin as far as the folds bounding the lateral recesses. Here they change their character and gradually become continuous with a thick layer of sensory epithelium which lines the internal surface of the pouches. At the folds which bound the median dorsal recess the sensory epithelium becomes continuous with hypodermic cells which line the folds, but at the top of the median recess there is a mass of ganglionic cells which differ from those composing the sensory epithelium. The hypodermic cells are large, rather columnar, and have a well-defined nucleus. The cells of the sensory epithelium are elongated and at the end next the chitin are prolonged into a thin projection of less diameter than the rest of the cell. The prolongation of each cell is just beneath the base of a chitinous hair, and I believe that the extreme point of each cell prolongation fits into a cavity in the base of a hair. The cells at the top of the median recess are elongated and in some cases have their internal end bifurcated. They are larger than the sense cells. The nerve supply is furnished by two short thick nerves which arise from the front part of the sub-œsophageal ganglion and pass to the posterior region of the organ, where they break up into fibres which supply the two divisions of the organ. This double nerve supply and the partially divided shape of the sac, already mentioned, prove in my opinion conclusively that the organ is a double one.

I believe this organ to be homologous with the tympanic organ of Insects. I regard the chitinous lining of each pouch as equivalent to the tympanic membrane of the insect auditory organ. In both cases we find a nerve breaking up into fibres which are connected with terminal structures which terminate in depressions in the chitin. V. Hensen regards the terminal structures or auditory rods (Hörstifte) in the insect organ as homologous with the hairs in the crustacean auditory sac. If his arguments hold good I think the chitinous hairs of the organ I have just described may be compared to both the structures just mentioned. From all that I have said it follows that I regard the organ of Scutigera as belonging to the class of organs usually described as auditory.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

February 2nd, 1885.

Prof. FOSTER, PRESIDENT, IN THE CHAIR.

Prof. C. S. Roy, M.A., was elected a Fellow.

The following communications were made to the Society:—

(1) *On the zeta-function in elliptic functions.* By J. W. L. GLAISHER, M.A., F.R.S.

The author considered three functions $\mathrm{ez}\,x$, $\mathrm{iz}\,x$, $\mathrm{gz}\,x$, defined by the equations

$$\mathrm{ez}\,x = \frac{E}{K}x + Z(x),$$

$$\mathrm{iz}\,x = \frac{I}{K}x + Z(x),$$

$$\mathrm{gz}\,x = \frac{G}{K}x + Z(x),$$

where

$$E = E \qquad = I + K,$$
$$I = E - K \quad = I,$$
$$G = E - k'^2 K = I + k^2 K,$$

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and $Z(x)$ has the meaning assigned to it by Jacobi. Denoting by accented letters the same functions of k' that the unaccented letters are of k, it can be shewn that

$$EK' + I'K = \tfrac{1}{2}\pi,$$
$$IK' + E'K = \tfrac{1}{2}\pi,$$
$$GK' + G'K = \tfrac{1}{2}\pi,$$

and

$$\mathrm{ez}\,(x + 2mK + 2m'iK') = \mathrm{ez}\,x + 2mE - 2m'iI',$$
$$\mathrm{iz}\,(x + 2mK + 2m'iK') = \mathrm{iz}\,x + 2mI - 2m'iE',$$
$$\mathrm{gz}\,(x + 2mK + 2m'iK') = \mathrm{gz}\,x + 2mG - 2m'iG'.$$

The function $\mathrm{ez}\,x$ is the same as Jacobi's $E(x)$, and $\mathrm{iz}\,x$ is the same as Weierstrass's function $\frac{\mathrm{Al}'x}{\mathrm{Al}\,x}$. The functions $\mathrm{ez}\,x$ and $\mathrm{iz}\,x$ form a pair of corresponding functions in which E and I, and E' and I', are interchanged, but $\mathrm{gz}\,x$ stands by itself and is such that corresponding to an increase of argument $2K$ the increase of the function is $2G$, and corresponding to an increase of $2iK'$ the increase of the function is $-2iG'$.

Three other functions $\mathrm{uz}\,x$, $\mathrm{vz}\,x$, $\mathrm{wz}\,x$, defined by the equations

$$\mathrm{uz}\,x = \frac{U}{K}x + Z(x),$$
$$\mathrm{vz}\,x = \frac{V}{K}x + Z(x),$$
$$\mathrm{wz}\,x = \frac{W}{K}x + Z(x),$$

where

$$U = E - \tfrac{1}{2}(1 + k'^2)K = I + \tfrac{1}{2}k^2K,$$
$$V = E - \tfrac{1}{2}k'^2K = I + \tfrac{1}{2}(1 + k^2)K,$$
$$W = E - \tfrac{1}{2}K = I + \tfrac{1}{2}K,$$

were also considered. These functions correspond exactly to $\mathrm{ez}\,x$, $\mathrm{iz}\,x$, $\mathrm{gz}\,x$, the quantities E, I, G, E', I', G' being replaced by U, V, W, U', V', W' respectively. There is thus a reciprocity between $\mathrm{uz}\,x$ and $\mathrm{vz}\,x$; but $\mathrm{wz}\,x$, like $\mathrm{gz}\,x$, is complete in itself. The general form of Zeta-function, and the systems of Theta-functions derived from these six Zeta-functions (corresponding to Weierstrass's $\mathrm{Al}x$, derived from $\mathrm{iz}\,x$), were also referred to. Those derived from $\mathrm{gz}\,x$ and $\mathrm{wz}\,x$ are the most complete and symmetrical.

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(2) *On a certain atomic hypothesis.* By Prof. K. PEARSON, communicated by H. T. STEARN, M.A.

[*Abstract.*]

THE paper deals with the motion of a number of spherical portions differentiated off from a continuous medium and pulsating uniformly over their surfaces. Each of these spheres, called in the paper an atom, has a natural period of free pulsation, depending on its mass and the potential energy it is supposed to possess. This natural period is modified by the presence of other atoms of the same or different period, and a form of expression is found for the resulting period of a system of a number of atoms exerting mutual influence on each other. Such a system is called a molecule; the atoms of one molecule are supposed not to affect the period of vibration of another molecule, the average distance between two molecules being great compared with that between two atoms.

Expressions are found for the forces arising from the motion:

(α) between two atoms in the same molecule, called chemical forces;

(β) between two molecules, containing the same or different atoms, in proximity to each other—molecular forces;

(γ) between groups of molecules at some distances apart.

The results of the mathematical investigation are applied to some of the phenomena of spectrum analysis, e.g. the relation between the spectra of some of the metalloids and their atomic weights, the effect of pressure in modifying the spectrum of a gas, and the relation between the spectrum of a compound and those of its constituents.

The paper is being published in full in the Transactions of the Society.

(3) *On a Young's Eriometer.* By R. T. GLAZEBROOK, M.A., F.R.S.

The author exhibited and described an "eriometer," an instrument for measuring the diameter of fine fibres, and so testing the value of different qualities of wool, which belonged to Thomas Young, and on which there was a description in his handwriting.

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February 16, 1885.

Prof. FOSTER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows:

J. H. Randell, B.A.
W. Gardiner, B.A.
R. Threlfall, B.A.

The following communications were made to the Society:—

(1) *Some remarks on the Urea-ferment.* By A. S. LEA, M.A.

The object of this communication was to demonstrate more definitely than has hitherto been done, the isolability of an amorphous, unorganised ferment, capable of rapidly effecting the ammonic conversion of urea from the Torula ureae.

Experiments were shewn demonstrating the isolation of the ferment and its dependence upon the Torula to which the fermentation was initially due.

(2) *On the occurrence of reproductive organs on the root of Laminaria bulbosa.* By W. GARDINER, B.A.

Having briefly dwelt upon Sach's idea of "root" and "shoot," the author stated that one of the fundamental ideas connected with root structures was that they did not bear reproductive organs. In *Laminaria bulbosa* however he found that sporangia are produced on the root and even at the very apex of the rootlets. In his opinion this occurrence was an example of "adaptation to environment," and did not therefore overthrow the usually received notions as expressed by Sachs. So far as he had observed, it was only when the frond had been broken off from the root by the violence of the waves or other causes, that the sporangia made their appearance.

(3) *On a new form of sporangium in Alaria esculenta, with suggestions as to the existence of sexual reproduction in the Laminaria.* By W. GARDINER, B.A.

So far the form of sporangium observed in Alaria was one containing four spores, usually known as tetraspores, and therefore asexual. The author has discovered another form which contained

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numerous small spores, each provided with cilia. On this account, and recalling the similarity of the Laminarias to the Fucaceae rather than to the Florideae, it appears to him probable that the so-called tetrasporangium containing four tetraspores is really a four-oosphered oogenium similar to the like structures occurring in certain of the Fucaceae, and that the sporangium discovered by himself is an antheridium containing antheroyoids. Thus here a sexual process is present which he believes also occurs generally in the Laminarias. A striking confirmation of this theory has been afforded by the phenomena occurring in the life-history of *D'Urvillea utilis.*

(4) *On the types of excretory system found in the Enteropneusta.* By W. Bateson, B.A.

The author described the development of the so-called "heart" of the Enteropneusta as a segregation of cells from the posterior wall of the anterior mesoblastic pouch, dorsal to the notochord. Between this structure and the notochord the true heart arises as a mesoblastic split. It gives off a plexus of vessels which are covered by conical cells, attached by their apices, in which the nuclei occur. These cells are similar in character to those lining the so-called "heart." The vascular plexus is differently arranged in different species. In Balanoglossus Kowalevskii the capillaries are irregular and anastomosing, while in B. salmoneus they are parallel and do not anastomose. In the plexus in B. Brooksii (new species) the condition is intermediate. An epiblastic ciliated sac is then formed in the skin on the left side of the proboscis stalk. After a time this sac communicates with the exterior and with the anterior body-cavity. It was suggested that this pore is excretory; and that certain yellowish-brown bodies found in and around the glandular cells are conveyed to the exterior by it.

A similar suggestion was offered as to the function of the two ciliated pores which open from the second body-cavity to the atrium. These pores arise as perforations through the outer wall of the atrium. In the second body-cavity brown bodies are found, similar to those in the proboscis-cavity, which are possibly carried out by these pores.

No evidence could be obtained of any currents flowing inwards at any of these pores.

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March 2, 1885.

Prof. FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:—

(1) *On some theorems in Tides and Long-waves.* By Rev. E. HILL, M.A.

Elementary considerations were given from which it might be inferred that when a disturbing body produces a semi-diurnal tide in an equatorial canal, the point nearest to the disturbing body will be a point of low tide or high tide according to the depth of the canal.

A general explanation was given of the influence of the depth of a canal on the speed of a long-wave traversing it.

It was shewn that the ordinary formula for this speed might be deduced from the ordinary differential equation of motion without integration.

(2) *On the electrical resistance of Platinum at high temperatures.* By W. N. SHAW, M.A.

(3) *On an automatic mechanical arrangement for maintaining a constant high potential.* By R. THRELFALL, B.A.

In connection with certain experiments which I wished to undertake some time ago it became desirable to obtain an apparatus capable of maintaining a condenser at a high potential for a considerable period. Although further consideration led me to abandon the experiments referred to, pending an enquiry into certain preliminary questions, I am of opinion that the arrangement I devised at the time for maintaining a high potential, may be of some independent interest, and this the more since I believe that there are many experiments which would be greatly facilitated if there was a ready method of maintaining condensers at a constant mean potential during considerable periods. Where the leakage is very small, as in Sir William Thomson's electrometers, much satisfaction may doubtless be obtained by the hand use of the replenishing guage at short intervals of time. There are cases however in which the fall of potential may be very much more rapid than in these instruments; and it is such cases that the arrangement to be described has been arranged to meet.

In the *Proceedings of the R. Society* for 1881 Lord Rayleigh has shewn how a water motor of the Thirlmere type may be arranged to run at an approximately constant speed; viz. by

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making the work which the motor has to do proportional to a high power of the speed. This is most easily obtained by setting the motor to drive a fan. If then we arrange a motor with a constant head of water in connection with a fan, we can by adjusting the water supply, and the dimensions of the fan, obtain a tolerably wide range over which the speed can be made constant at any desired point. Let a motor and fan therefore be fitted up: and thus governed let it drive the arrangement shewn in the diagrammatic drawing.

A copper disc is placed between the poles of an electromagnet, this disc is made to rotate with a tolerably high velocity by means of the governed motor. As long as no current is flowing round the magnet the energy spent in rotating the disc is very small.

In the circuit there is mica or glass condenser of considerable capacity; its actual dimensions and construction depend on the potential at which the apparatus is desired to be maintained, and on its capacity. If the capacity of the apparatus is itself large then no condenser will be necessary. There is a pully which is driven by a much smaller one on the same shaft as the copper disc, and is connected with a replenishing guage of a type to be determined by the special conditions of the experimental apparatus. For instance, the value of the desired potential will determine what precautions must be taken in insulating and this will determine the size of the replenisher. The electrical output required to compensate the fall of potential which it is desired to guard against will determine the further construction of the replenisher.

A fixed disc is in connection with that pole of the apparatus which it is desired to maintain at a constant high potential, another disc is suspended by a long and fine spiral spring which in its turn is hung to a wire the other end of which goes to a Weber suspension mounted on the top of the guard tube, and not shewn in the diagram. The usual method of levelling the suspended plate is adopted. Since there is no need for any particular accuracy of levelling this may be easily accomplished. The suspended plate carries a wire whose ends are bent down vertically and tipped with aluminium. Below the ends of these wires are placed two mercury cups on adjustable stands. The object of using aluminium joints is to avoid the capillary effects so noticeable in the case of platinum or any amalgamated contact breakers. Mercury is generally stated to wet aluminium without amalgamating with it, but from some experiments which I made on the point I have concluded that it can only be with certain specimens of aluminium that this takes place; with ordinary foil and wire I invariably found that the mercury is always on the point of wetting the aluminium but never quite succeeds. The advantage

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of this is that the mercury will not cling to the points while they are rising from it; and that therefore the beat will be much sharper than in the case of platinum where an increase of resistance must precede the break of contact. For many purposes this is an advantage as it diminishes the intensity of the self induction spark. Since however in this apparatus the efficiency of the regulator depends on the definiteness of the point of contact, anything tending to diminish this must be more or less injurious. Since this was written I have discovered that the aluminium burns away so rapidly that, balancing disadvantages, platinum is on the whole to be preferred.

The action of the regulator will now be easily understood. A battery continually maintains a current round the electro-magnet: and through a high resistance inserted in circuit between the battery and the magnet.

When the potential rises above the desired point, the attracted disc by means of its contact makes short-circuits the high resistance and allows an increased current to flow round the electro-magnet. This will tend to put an increased brake on the copper disc and the rotating parts of the apparatus will instantly slow down. I say instantly, because if india-rubber bands be used to transmit the motion between the various pullies we shall only have to take into account the momentum of the copper disc and replenisher and not that of the motor: at least during short intervals of time. Now the copper disc may be made pretty thin and the other parts are not very large, so that a practically instantaneous change of velocity may be obtained: it is only a question of having a large enough battery—say two Groves', a fairly high resistance to be short-circuited, and long india-rubber bands.

For a given change of potential the attracted disc will be more sensitive the nearer it is to the fixed disc. The method of using the regulating arrangement is therefore as follows. The apparatus to be maintained at the constant high potential is charged up to this potential as indicated say by a portable electrometer; and by means of the Weber suspension and the adjustable support of the mercury cups the attracted plate is lowered so as to make contact when the potential is slightly increased. The limit of the sensitiveness of the apparatus depends on the critical distance of the plates, and this depends on the law of stretching of the spring. As a general rule in a long spring the extension will not be far from proportional to the stretching force. Sir William Thomson is in the habit of using the torsion of a wire in some of his instruments instead of a spring. In this case however we wish to make the instrument as sensitive as possible always, and not as in electrometers so as to have a constant relative sensitiveness: we must therefore have an adjustment for working at different distances between the

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plates, and this, I take it, is most easily accomplished by a spring hung from a Weber suspension in a tube which will protect it from atmospheric disturbances.

When everything is properly adjusted the plate ought to settle down into a state of pretty rapid vibration, so that the mean potential difference say during each minute is constant. I had at one time intended to make the movable disc insert or remove a varying resistance; as for instance by making the contact points conical. There were however many mechanical difficulties: and worse than that the constancy of the battery would have required attention in any continuous regulator. Moreover unless a spring of peculiar construction was used the upper plate would be unstable, and this would be fatal. The only way of guarding against it would be by arranging the ratios of the resistances taken in and out; but this would have been impossible even supposing the current was not strong enough to heat the wires so as to produce a continuous change. For all practical purposes however a constant integral value of the potential is all that is required, and this may be obtained from a discontinuous governor. The drawings are not to scale, because the scale as already pointed out will depend on the purpose for which the governor is to be used.

March 16, 1885.

Prof. FOSTER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows:

Prof. J. J. Thomson, M.A.
J. M. Dodds, M.A.
Rev. A. H. Cooke, M.A.
A. E. Shipley, B.A.

The following communications were made to the Society:

(1) *Further remarks on the Urea-ferment.* By A. S. LEA, M.A.

In the previous communication made to the Society it was stated that although the Torula ureae contains a ferment which is soluble in water, this ferment is not to be found in the fluid surrounding the cells either during or after an active alkaline fermentation. It appears therefore that the cells do not carry on the fermentation by excreting the ferment with the surrounding fluid, but that its activity is localised in the substance of the cells. The absence of the ferment in the fluid during an active fermentation appeared at first sight somewhat unexpected, in view of

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the fact that it is soluble, and it was hence a matter of interest to see whether, in the case of other ferment actions produced by similar organisms, also containing a soluble ferment, this ferment is similarly retained within the cell protoplasm. Working with ordinary yeast (Torula cerevisiae) the author finds that at no time during an active fermentation is any invertin to be found in the fluid surrounding the cells. In this case as in the former the ferment is retained by the cells. Experiments were described in proof of the above statement. The invertin is not used up during the change which it produces, neither do the yeast-cells destroy it during their growth and activity. This is shewn by the facts that if cane-sugar be inverted by a solution of invertin, this same solution is again capable of inverting a further portion of cane-sugar, and that if some invertin be added to yeast, and an active fermentation then carried on with this yeast, then the invertin can be recovered from the fluid surrounding the cells.

It was suggested that the retention of the ferment by the cells in the above cases is due to the fact that, unlike animal cells, the Torulae are inclosed by a complete cellulose envelope which is not broken up at the death of the cell, and that since all known ferments though soluble are not diffusible, therefore even after the death of the cell the ferment is retained in the dead cell-substance. If the cells are treated in such a way as to break up the cellulose envelope then the soluble ferments are capable of being extracted with water.

The outcome of the experiments shews that the ferment action of these organisms is carried on in their protoplasm and not in the fluid surrounding them by the excretion of every soluble ferment.

(2) *On some points in the anatomy of Nebalia.* By G. WELDON, M.A.

(3) *Observations on the constitution of Callus.* By WALTER GARDINER, B.A.

The author remarked that perhaps no structure had given rise to so many differences of opinion as the callus of sieve-tubes. So far as the question of its derivation was concerned there could be little doubt that the views of Russow and Strasburger, who regarded it as being derived from the protoplasm, were right, and that Wilhelm and Janczewski's view (as to its derivation from the mucilaginous degeneration of the cell-wall) could not be maintained. While from its high organisation it seemed almost imperative to regard it as being formed in connection with, and through the agency of, living protoplasm, yet the very early disappearance of the nucleus and the scant quantity of the dying protoplasmic contents threw some doubt upon this assumption.

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The microchemical characteristics were so peculiar as to be quite unique. Taking into consideration, however, some of these reactions (especially in the light of the results obtained with Hofmann's blue and Szyszylowicz's corallin-soda): the presence of, and the gradual breaking down of, the sieve-tube starch grains and the mucilaginous degeneration of the protoplasmic contents: there seemed to be little doubt that the callus is in reality of the nature of a starchy mucilage. Such a body would give all the reactions of callus. The author then drew attention to the wide occurrence of the callus in connection with living cells, and remarked that from his observations he was led to believe that a callus may also occur on the pits of young tracheids, e.g. the root of Sambucus and the stem of Phaseolus.

(4) *Observations on Vegetable Proteids.* By J. R. GREEN, B.A.

The investigation of which the paper was a summary was undertaken first to determine the proteid constituents of the calx of the Cow-tree of South America (*Brosimum galactodendron*).

To this calx the name of vegetable milk has been given; it is described as closely resembling cow's milk in appearance, but having a slightly resinous odour. It contains two proteids, one being of the nature of a true albumin, the other belonging to the group of peptones.

The former is soluble in water, coagulates at a temperature of 68° C., and gives a well-marked xanthoproteic reaction.

The latter of the two is the body which formed the subject of a communication to the Society at the end of the October Term, 1884.

In the calx of the Balata tree (*Mimusops globosa*) a globulin body occurs which is absent from Brosimum. It is marked by the usual reactions of globulins, but presents some features of interest in connection with the effect of heat upon it. Its coagulating point varies with the reaction of the liquid in which it is dissolved. In an acid liquid opalescence sets in at 75°—80°C., and as it cools a precipitate settles out. This is insoluble in nitric acid, but dissolves to a large extent in ammonia.

In an alkaline solution the coagulating point is 79°—85° C., and the resulting precipitate is soluble in nitric acid but not in alkalis.

The proteid bodies in both these fluids were found to be partly destroyed by the spirit which was mixed with the calx for purposes of preservation. Hence it seemed well to examine some fresh plants. Those selected were *Manihot glaziovii*, the common lettuce, and the cabbage; these being representatives of three natural orders not closely allied to each other.

In the manihot a globulin exists with well-marked characters. It is precipitated from solution by a current of CO_2 or by satu-

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ration with solid $MgSO_4$. It gives well-marked xanthoproteic reaction, and coagulates at a temperature of 74°—80° C.

In the lettuce is a body closely resembling the *Hemialbumose* described by Dr Vines as occurring in many seeds.

It is precipitated from its solutions by nitric or acetic acids, and the solution of the precipitate in water does not coagulate on boiling. It is precipitated also by potassic ferrocyanide and acetic acid.

In all the plants above mentioned and in the calx of the gutta-percha tree of the East Indies the peculiar proteid alluded to above appears to be present.

In the vegetable kingdom thus representatives of the three great classes of proteids, albumins, globulins, and peptones are present.

(5) *On the developments of K', E', J', G' in ascending powers of the Modulus.* Part II. By J. W. L. GLAISHER, M.A.

The present paper is a continuation of one bearing the same title which was read to the Society on November 24, 1884, and was published on pp. 184—208 of this volume. The sections are numbered consecutively with those of the former paper.

Forms of the equation $EK' + E'K - KK' = \frac{1}{2}\pi$, §§ 31, 32.

§ 31. The expression

$$EK' + E'K - KK'$$

may evidently be written in the forms

(i) $EK' + (E' - K')\,K$,
(ii) $(E - K)\,K' + E'K$,
(iii) $(E - k'^2K)\,K' + (E' - k^2K')\,K$,
(iv) $\{E - \frac{1}{2}(1 + k'^2)K\}\,K' + (E' - \frac{1}{2}k^2K')\,K$,
(v) $(E - \frac{1}{2}k'^2K)\,K' + \{E' - \frac{1}{2}(1 + k^2)\,K'\}\,K$,
(vi) $(E - \frac{1}{2}K)K' + (E' - \frac{1}{2}K')\,K$;

that is, in the forms

(i) $EK' + I'K$,
(ii) $IK' + E'K$,
(iii) $GK' + G'K$,
(iv) $UK' + V'K$,
(v) $VK' + U'K$,
(vi) $WK' + W'K$.

We thus deduce from the well-known formula

$$EK' + E'K - KK' = \tfrac{1}{2}\pi \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

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the six equations

$$\text{(i)} \quad EK' + I'K = \tfrac{1}{2}\pi,$$
$$\text{(ii)} \quad IK' + E'K = \tfrac{1}{2}\pi,$$
$$\text{(iii)} \quad GK' + G'K = \tfrac{1}{2}\pi,$$
$$\text{(iv)} \quad UK' + V'K = \tfrac{1}{2}\pi,$$
$$\text{(v)} \quad VK' + U'K = \tfrac{1}{2}\pi,$$
$$\text{(vi)} \quad WK' + W'K = \tfrac{1}{2}\pi.$$

Each of these equations is thus a form of (1), the three terms being reduced to two. The last three equations are immediately deducible from the first three by addition, since

$$U = \tfrac{1}{2}(I + G), \quad V = \tfrac{1}{2}(G + E), \quad W = \tfrac{1}{2}(E + I).$$

Equations (iii) and (vi), which involve G and W, are perfectly symmetrical. The other four formulæ exhibit the correspondence between E and I, and between U and V, which is observable in so many other results.

Dividing by KK', the six equations become

$$\text{(i)} \quad \frac{E}{K} + \frac{I'}{K'} = \frac{\pi}{2KK'},$$
$$\text{(ii)} \quad \frac{I}{K} + \frac{E'}{K'} = \frac{\pi}{2KK'},$$
$$\text{(iii)} \quad \frac{G}{K} + \frac{G'}{K'} = \frac{\pi}{2KK'},$$
$$\text{(iv)} \quad \frac{U}{K} + \frac{V'}{K'} = \frac{\pi}{2KK'},$$
$$\text{(v)} \quad \frac{V}{K} + \frac{U'}{K'} = \frac{\pi}{2KK'},$$
$$\text{(vi)} \quad \frac{W}{K} + \frac{W'}{K'} = \frac{\pi}{2KK'}.$$

§ 32. If we express the relation (1) in terms of K and one other letter, the expression equated to $\frac{1}{2}\pi$ contains three terms except when the second letter is G or W.

The group of four formulæ to which (1) belongs is

$$(1) \quad EK' + E'K - KK' = \tfrac{1}{2}\pi,$$
$$(2) \quad IK' + I'K + KK' = \tfrac{1}{2}\pi,$$
$$(3) \quad UK' + U'K + \tfrac{1}{2}KK' = \tfrac{1}{2}\pi,$$
$$(4) \quad VK' + V'K - \tfrac{1}{2}KK' = \tfrac{1}{2}\pi.$$

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The additional term arises from the fact that it is not E and E', I and I', U and U', and V and V' that correspond to each other, but E and I', I and E', U and V', and V and U'. The quantities G and G' correspond to each other, as also do W and W'.

Similar formulæ in which K and K' are not involved. §§ 33, 34.

§ 33. The following fifteen equations in which K and K' are not involved are also forms of the relation (i)

$$\text{(vii)} \quad EE' - II' = \tfrac{1}{2}\pi,$$
$$\text{(viii)} \quad VV' - UU' = \tfrac{1}{4}\pi,$$
$$\text{(ix)} \quad GW' - G'W = \tfrac{1}{4}(k^2 - k'^2)\pi,$$
$$\text{(x)} \quad E'G - IG' = \tfrac{1}{2}k'^2\pi,$$
$$\text{(xi)} \quad EG' - IG' = \tfrac{1}{2}k^2\pi,$$
$$\text{(xii)} \quad VW' - U'W = \tfrac{1}{4}k^2\pi,$$
$$\text{(xiii)} \quad V'W - UW' = \tfrac{1}{4}k'^2\pi,$$
$$\text{(xiv)} \quad EW' - I'W = \tfrac{1}{4}\pi,$$
$$\text{(xv)} \quad E'W - IW' = \tfrac{1}{4}\pi,$$
$$\text{(xvi)} \quad VG' - U'G = \tfrac{1}{4}k'^2\pi,$$
$$\text{(xvii)} \quad V'G - UG' = \tfrac{1}{4}k^2\pi,$$
$$\text{(xviii)} \quad EU' - I'V = \tfrac{1}{4}k'^2\pi,$$
$$\text{(xix)} \quad E'U - IV' = \tfrac{1}{4}k^2\pi,$$
$$\text{(xx)} \quad EV' - I'U = \tfrac{1}{4}(1 + k'^2)\pi,$$
$$\text{(xxi)} \quad E'V - IU' = \tfrac{1}{4}(1 + k^2)\pi.$$

The first three equations remain unaltered when k^2 is changed into k'^2, but the other twelve equations form six pairs, the equations in each pair being convertible into each other by this change.

These fifteen equations may be readily verified by expressing all the quantities in terms of K and K' and any other two quantities which correspond to each other. Taking for example E and I' as the two corresponding quantities all the equations reduce to (i) of § 31, viz. to

$$EK' + I'K = \tfrac{1}{2}\pi,$$

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if we substitute for the quantities involved the values assigned to them by the equations:

$$\begin{aligned} E &= E, & E' &= I' + K', \\ I &= E - K, & I' &= I', \\ G &= E - k'^2 K, & G' &= I' + k'^2 K', \\ U &= E - \tfrac{1}{2}(1 + k'^2) K, & U' &= I' + \tfrac{1}{2} k'^2 K', \\ V &= E - \tfrac{1}{2} k'^2 K, & V' &= I' + \tfrac{1}{2}(1 + k'^2) K', \\ W &= E - \tfrac{1}{2} K, & W' &= I' + \tfrac{1}{2} K'. \end{aligned}$$

Thus, for example, equation (xx) becomes

$$E\{I' + \tfrac{1}{2}(1 + k'^2) K'\} - I' \{E - \tfrac{1}{2}(1 + k'^2)\} K = \tfrac{1}{4}(1 + k'^2) \pi,$$

that is

$$\tfrac{1}{2}(1 + k'^2)(EK' + I'K) = \tfrac{1}{4}(1 + k'^2)\pi,$$

which is equivalent to (i).

The systems of formulæ which express E, I, G, U, V, W in terms of any one of them and K are given in § 45.

§ 34. It will be noticed that in the 21 formulæ (i)...(xxi) each letter occurs in combination with every other letter except the one to which it corresponds. Taking, for example, E we find the combinations

$$EK',\ EE',\ EG',\ EU',\ EV',\ EW',$$

but not EI'. Similarly we find

$$GK',\ GE',\ GI',\ GU',\ GV',\ GW'.$$

but not GG'.

Relations involving K_1, K', E_1, E', &c., § 35.

§ 35. Using K_1, E_1, I_1, &c. as in § 9 (p. 191), to denote $\frac{2K}{\pi}$, $\frac{2E}{\pi}$, $\frac{2I}{\pi}$, &c., the 21 equations may be written:

$$\begin{aligned} &\text{(i)} && E_1K' + I'K_1 = 1, \\ &\text{(ii)} && I_1K' + E'K_1 = 1, \\ &\text{(iii)} && G_1K' + G'K_1 = 1, \\ &\text{(iv)} && U_1K' + V'K_1 = 1, \\ &\text{(v)} && V_1K' + U'K_1 = 1, \\ &\text{(vi)} && W_1K' + W'K_1 = 1, \\ &\text{(vii)} && E_1E' - I_1I' = 1, \end{aligned}$$

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$$
\begin{aligned}
&\text{(viii)} && V_1V' - U_1U' = \tfrac{1}{2},\\
&\text{(ix)} && G_1W' - G'W_1 = \tfrac{1}{2}(k^2 - k'^2),\\
&\text{(x)} && E_1G' - I'G_1 = k'^2,\\
&\text{(xi)} && E'G_1 - I_1G' = k^2,\\
&\text{(xii)} && V_1W' - U'W_1 = \tfrac{1}{2}k^2,\\
&\text{(xiii)} && V'W_1 - U_1W' = \tfrac{1}{2}k'^2,\\
&\text{(xiv)} && E_1W' - I'W_1 = \tfrac{1}{2},\\
&\text{(xv)} && E'W_1 - I_1W' = \tfrac{1}{2},\\
&\text{(xvi)} && V_1G' - U'G_1 = \tfrac{1}{2}k'^2,\\
&\text{(xvii)} && V'G_1 - U_1G' = \tfrac{1}{2}k^2,\\
&\text{(xviii)} && E_1U' - I'V_1 = \tfrac{1}{2}k'^2,\\
&\text{(xix)} && E'U_1 - I_1V' = \tfrac{1}{2}k^2,\\
&\text{(xx)} && E_1V' - I'U_1 = \tfrac{1}{2}(1 + k'^2),\\
&\text{(xxi)} && E'V_1 - I_1U' = \tfrac{1}{2}(1 + k^2).
\end{aligned}
$$

Relations involving K_1, K_2, E_1, E_2, *&c.*, §§ 36—38.

§ 36. By § 10 (p. 193) we have

$$K' = K_1 \log\frac{4}{k} - K_2,$$

$$E' = -I_1 \log\frac{4}{k} + I_2,$$

$$I' = -E_1 \log\frac{4}{k} + E_2,$$

$$G' = -G_1 \log\frac{4}{k} + G_2,$$

$$U' = -V_1 \log\frac{4}{k} + V_2,$$

$$V' = -U_1 \log\frac{4}{k} + U_2,$$

$$W' = -W_1 \log\frac{4}{k} + W_2.$$

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Substituting these values of K', E', &c. in the equations given in the last section we find that in every case the terms multiplied by $\log \frac{4}{k}$ cancel each other, and we obtain the following 21 equations involving K_1, K_2, E_1, E_2, &c.:

$$\begin{aligned}
&\text{(i)} && E_2K_1 - E_1K_2 = 1,\\
&\text{(ii)} && I_2K_1 - I_1K_2 = 1,\\
&\text{(iii)} && G_2K_1 - G_1K_2 = 1,\\
&\text{(iv)} && U_2K_1 - U_1K_2 = 1,\\
&\text{(v)} && V_2K_1 - V_1K_2 = 1,\\
&\text{(vi)} && W_2K_1 - W_1K_2 = 1,\\
&\text{(vii)} && E_1I_2 - I_1E_2 = 1,\\
&\text{(viii)} && V_1U_2 - U_1V_2 = \tfrac{1}{2},\\
&\text{(ix)} && G_1W_2 - G_2W_1 = \tfrac{1}{2}(k^2 - k'^2),\\
&\text{(x)} && E_1G_2 - E_2G_1 = k'^2,\\
&\text{(xi)} && I_2G_1 - I_1G_2 = k^2,\\
&\text{(xii)} && V_1W_2 - V_2W_1 = \tfrac{1}{2}k^2,\\
&\text{(xiii)} && U_2W_1 - U_1W_2 = \tfrac{1}{2}k'^2,\\
&\text{(xiv)} && E_1W_2 - E_2W_1 = \tfrac{1}{2},\\
&\text{(xv)} && I_2W_1 - I_1W_2 = \tfrac{1}{2},\\
&\text{(xvi)} && V_1G_2 - V_2G_1 = \tfrac{1}{2}k'^2,\\
&\text{(xvii)} && U_2G_1 - U_1G_2 = \tfrac{1}{2}k^2,\\
&\text{(xviii)} && E_1V_2 - E_2V_1 = \tfrac{1}{2}k'^2,\\
&\text{(xix)} && I_2U_1 - I_1U_2 = \tfrac{1}{2}k^2,\\
&\text{(xx)} && E_1U_2 - E_2U_1 = \tfrac{1}{2}(1 + k'^2),\\
&\text{(xxi)} && I_2V_1 - I_1V_2 = \tfrac{1}{2}(1 + k^2).
\end{aligned}$$

§ 37. It will be seen that by the substitution for K', E', &c. of their values in terms of K_1 and K_2, I_1 and I_2, &c. all the 21 equations have been rendered uniform and symmetrical, viz. the sign of the second term is negative in the equations involving K_1 and K_2 as well as in the others, and each expression is symmetrical,

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i.e. the first term being of the form A_1B_2, the second is of the form A_2B_1.

In the following list the equations are written in a uniform manner and so arranged in groups as to exhibit the correspondence between E and V, I and U, and G and W.

$$
\begin{aligned}
&(1)\quad K_1E_2 - K_2E_1 = 1,\\
&(2)\quad K_1I_2 - K_2I_1 = 1,\\
&(3)\quad K_1G_2 - K_2G_1 = 1,\\
&(4)\quad K_1U_2 - K_2U_1 = 1,\\
&(5)\quad K_1V_2 - K_2V_1 = 1,\\
&(6)\quad K_1W_2 - K_2W_1 = 1,\\
&(7)\quad E_1I_2 - E_2I_1 = 1,\\
&(8)\quad E_1G_2 - E_2G_1 = k'^2,\\
&(9)\quad E_1U_2 - E_2U_1 = \tfrac{1}{2}(1+k'^2),\\
&(10)\quad E_1W_2 - E_2W_1 = \tfrac{1}{2},\\
&(11)\quad E_1V_2 - E_2V_1 = \tfrac{1}{2}k'^2,\\
&(12)\quad V_1I_2 - V_2I_1 = \tfrac{1}{2}(1+k^2),\\
&(13)\quad V_1G_2 - V_2G_1 = \tfrac{1}{2}k'^2,\\
&(14)\quad V_1U_2 - V_2U_1 = \tfrac{1}{2},\\
&(15)\quad V_1W_2 - V_2W_1 = \tfrac{1}{2}k^2,\\
&(16)\quad G_1I_2 - G_2I_1 = k^2,\\
&(17)\quad G_1U_2 - G_2U_1 = \tfrac{1}{2}k^2,\\
&(18)\quad G_1W_2 - G_2W_1 = \tfrac{1}{2}(k^2-k'^2),\\
&(19)\quad W_1I_2 - W_2I_1 = \tfrac{1}{2},\\
&(20)\quad W_1U_2 - W_2U_1 = \tfrac{1}{2}k'^2,\\
&(21)\quad U_1I_2 - U_2I_1 = \tfrac{1}{2}k^2.
\end{aligned}
$$

§ 38. The system of equations is perhaps best written in the following form, in which the left-hand members of the equations are expressed as determinants and the equations are so grouped as to exhibit more clearly than in the last section the correspondence between E and V, I and U, and G and W.

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$$\begin{vmatrix} K_1, & E_1 \\ K_2, & E_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} K_1, & V_1 \\ K_2, & V_2 \end{vmatrix} = 1,$$

$$\begin{vmatrix} K_1, & I_1 \\ K_2, & I_1 \end{vmatrix} = 1, \qquad \begin{vmatrix} K_1, & U_1 \\ K_2, & U_2 \end{vmatrix} = 1,$$

$$\begin{vmatrix} K_1, & G_1 \\ K_2, & G_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} K_1, & W_1 \\ K_2, & W_2 \end{vmatrix} = 1,$$

$$\begin{vmatrix} E_1, & I_1 \\ E_2, & I_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} V_1, & U_1 \\ V_2, & U_2 \end{vmatrix} = \tfrac{1}{2},$$

$$\begin{vmatrix} E_1, & G_1 \\ E_2, & G_2 \end{vmatrix} = k'^2, \qquad \begin{vmatrix} V_1, & W_1 \\ V_2, & W_2 \end{vmatrix} = \tfrac{1}{2}k^2,$$

$$\begin{vmatrix} E_1, & U_1 \\ E_2, & U_2 \end{vmatrix} = \tfrac{1}{2}(1+k'^2), \qquad \begin{vmatrix} V_1, & I_1 \\ V_2, & I_2 \end{vmatrix} = \tfrac{1}{2}(1+k^2),$$

$$\begin{vmatrix} E_1, & W_1 \\ E_1, & W_2 \end{vmatrix} = \tfrac{1}{2}, \qquad \begin{vmatrix} V_1, & G_1 \\ V_2, & G_2 \end{vmatrix} = \tfrac{1}{2}k'^2,$$

$$\begin{vmatrix} E_1, & V_1 \\ E_2, & V_2 \end{vmatrix} = \tfrac{1}{2}k'^2,$$

$$\begin{vmatrix} U_1, & I_1 \\ U_2, & I_2 \end{vmatrix} = \tfrac{1}{2}k^2,$$

$$\begin{vmatrix} G_1, & I_1 \\ G_2, & I_2 \end{vmatrix} = k^2, \qquad \begin{vmatrix} W_1, & U_1 \\ W_2, & U_2 \end{vmatrix} = \tfrac{1}{2}k'^2,$$

$$\begin{vmatrix} G_1, & U_1 \\ G_2, & U_2 \end{vmatrix} = \tfrac{1}{2}k^2, \qquad \begin{vmatrix} W_1, & I_1 \\ W_2, & I_2 \end{vmatrix} = \tfrac{1}{2},$$

$$\begin{vmatrix} G_1, & W_1 \\ G_2, & W_2 \end{vmatrix} = \tfrac{1}{2}(k^2 - k'^2).$$

There are seven letters involved, viz.

$$K,\ E,\ I,\ G,\ U,\ V,\ W$$

and in these 21 equations, each letter is combined with every letter except itself; so that every possible combination occurs.

Thus, writing for brevity the determinant

$$\begin{vmatrix} A_1, & B_1 \\ A_2, & B_2 \end{vmatrix} \text{ as } (AB),$$

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and taking any letter, say for example G, the equations give the values of

$$(GK),\ (GE),\ (GI),\ (GU),\ (GV),\ (GW).$$

Definition of the Adjunct, § 39.

§ 39. The series for $K_1, E_1, \ldots, 2V_1, 2W_1$ and $K_2, E_2, \ldots, 2V_2, 2W_2$ in ascending powers of k^2 were given on p. 192 (§ 9).

The coefficient of k^{2n} in any series A_2 is connected with the corresponding term in the series A_1 by the curious law explained in § 4 (p. 188). Thus, taking for example the letter E, the coefficient of k^6 in E_1 is

$$-\frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2},$$

and the coefficient of k^6 in E_2 is

$$-\frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2}\left(\frac{2}{1}-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{1}{5}-\frac{2}{6}\right).$$

The quantity in brackets may be conveniently termed the *adjunct* to the coefficient $\frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2}$ and the above expression may be written

$$-\frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2}\ (\text{ad}).$$

The series for K_1, E_1, &c., and K_2, E_2, &c., § 40.

§ 40. The series for $K_2, E_2, \ldots 2V_2, 2W_2$ are derivable from those for $K_1, E_1, \ldots 2V_1, 2W_1$ by appending to each term after the first its adjunct. The first or constant term is either 0, 1 or 2 and follows no regular law. The coefficients of k^2 in $2U_1$ and $2U_2$ are anomalous, being 0 and $-\frac{1}{2}$. With these exceptions, all the series with suffix 2 differ from the corresponding series with suffix 1 only by the addition of the adjunct to each coefficient.

Using the notation explained in the last section the fourteen series given on p. 192 may be written

$$K_1 = 1 + \frac{1^2}{2^2}\ k^2 + \frac{1^2 . 3^2}{2^2 . 4^2}\ k^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}\ k^6 + \&c.,$$

$$K_2 = \frac{1^2}{2^2}(\text{ad})\ k^2 + \frac{1^2 . 3^2}{2^2 . 4^2}\ (\text{ad})\ k^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}\ (\text{ad})\ k^6 + \&c.,$$

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$$E_1 = 1 - \frac{1}{2^2} \quad k^2 - \frac{1^2 . 3}{2^2 . 4^2} \quad k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2} \quad k^6 - \&c.,$$

$$E_2 = 1 - \frac{1}{2^2} (\text{ad})\, k^2 - \frac{1^2 . 3}{2^2 . 4^2} (\text{ad})\, k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2} (\text{ad})\, k^6 - \&c.,$$

$$I_1 = -\frac{1}{2} \quad k^2 - \frac{1^2 . 3}{2^2 . 4} \quad k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6} \quad k^6 - \&c.,$$

$$I_2 = 1 - \frac{1}{2} (\text{ad})\, k^2 - \frac{1^2 . 3}{2^2 . 4} (\text{ad})\, k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6} (\text{ad})\, k^6 - \&c.,$$

$$G_1 = \frac{1^0}{2} \quad k^2 + \frac{1^2}{2^2 . 4} \quad k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6} \quad k^6 + \&c.,$$

$$G_2 = 1 + \frac{1^0}{2} (\text{ad})\, k^2 + \frac{1^2}{2^2 . 4} (\text{ad})\, k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6} (\text{ad})\, k^6 + \&c.,$$

$$2U_1 = -\frac{1^2}{2 . 4} \quad k^4 - \frac{1^2 . 3^2}{2^2 . 4 . 6} \quad k^6 - \&c.,$$

$$2U_2 = 2 - \tfrac{1}{2} k^2 - \frac{1^2}{2 . 4} (\text{ad})\, k^4 - \frac{1^2 . 3^2}{2^2 . 4 . 6} (\text{ad})\, k^6 - \&c.,$$

$$2V_1 = 1 + \frac{1^0}{2^2} \quad k^2 + \frac{1^2}{2^2 . 4^2} \quad k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6^2} \quad k^6 + \&c.,$$

$$2V_2 = 2 + \frac{1^0}{2^2} (\text{ad})\, k^2 + \frac{1^2}{2^2 . 4^2} (\text{ad})\, k^4 + \frac{1^2 . 3^2}{2^2 . 4^2 . 6^2} (\text{ad})\, k^6 + \&c.,$$

$$2W_1 = 1 - \frac{1.3}{2^2} \quad k^2 - \frac{1^2 . 3 . 5}{2^2 . 4^2} \quad k^4 - \frac{1^2 . 3^2 . 5 . 7}{2^2 . 4^2 . 6^2} \quad k^6 - \&c.,$$

$$2W_2 = 2 - \frac{1.3}{2^2} (\text{ad})\, k^2 - \frac{1^2 . 3 . 5}{2^2 . 4^2} (\text{ad})\, k^4 - \frac{1^2 . 3^2 . 5 . 7}{2^2 . 4^2 . 6^2} (\text{ad})\, k^6 - \&c.$$

The 21 relations connecting the 14 series, § 41.

§ 41. The equations in § 38 form a very remarkable series of relations to which these 14 series are subject, viz. denoting, as in that section, $A_1 B_2 - A_2 B_1$ by (AB), the 14 series are connected by the 21 equations:

$$(K, E) = 1, \qquad (K, 2V) = 2,$$
$$(K, I) = 1, \qquad (K, 2U) = 2,$$
$$(K, G) = 1, \qquad (K, 2W) = 2,$$

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$$\begin{aligned}
(E,\ I) &= 1, & (2V,\ 2U) &= 2,\\
(E,\ G) &= 1-k^2, & (2V,\ 2W) &= 2k^2,\\
(E,\ 2U) &= 2-k^2, & (2V,\ I) &= 1+k^2,\\
(E,\ 2W) &= 1, & (2V,\ G) &= 1-k^2,
\end{aligned}$$

$$(E,\ 2V) = 1-k^2,$$

$$(2U,\ I) = k^2,$$

$$\begin{aligned}
(G,\ I) &= k^2, & (2W,\ 2U) &= 2-2k^2,\\
(G,\ 2U) &= k^2, & (2W,\ I) &= 1,
\end{aligned}$$

$$(G,\ 2W) = 2k^2-1.$$

Thus, if A_1 and B_1 denote any two different series of the seven series

$$K_1,\ E_1,\ I_1,\ G_1,\ 2U_1,\ 2V_1,\ 2W_1,$$

then $A_1B_2 - A_2B_1$ is always of the form $a+bk^2$, where a and b have only the values 0, 1, 2.

For example, the first equation $K_1E_2 - K_2E_1 = 1$, is equivalent to the identity:

$$\begin{aligned}
&\left\{1+\frac{1^2}{2^2}\quad k^2+\frac{1^2.3^2}{2^2.4^2}\quad k^4+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\quad k^6+\&c.\right\}\\
&\times\left\{1-\frac{1}{2^2}(\mathrm{ad})\,k^2-\frac{1^2.3}{2^2.4^2}(\mathrm{ad})\,k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}(\mathrm{ad})\,k^6-\&c.\right\}\\
&-\left\{1-\frac{1}{2^2}\quad k^2-\frac{1^2.3}{2^2.4^2}\quad k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}\quad k^6-\&c.\right\}\\
&\times\left\{\quad\frac{1^2}{2^2}(\mathrm{ad})\,k^2+\frac{1^2.3^2}{2^2.4^2}(\mathrm{ad})\,k^4+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}(\mathrm{ad})\,k^6+\&c.\right\}=1.
\end{aligned}$$

Arithmetical formulæ derived from the 21 *relations*, §§ 42—45.

§ 42. The general form of the 21 relations is

$$\begin{aligned}
&(\alpha\ +P_1\ k^2+P_2\ k^4+P_3\ k^6+P_4\ k^8+\&c.)\\
&\times(\beta_0+Q_1q_1k^2+Q_2q_2k^4+Q_3q_3k^6+Q_4q_4k^8+\&c.)\\
&-(\beta\ +Q_1\ k^2+Q_2\ k^4+Q_3\ k^6+Q_4\ k^8+\&c.)\\
&\times(\alpha_0+P_1p_1k^2+P_2p_2k^4+P_3p_3k^6+P_4p_4k^8+\&c.)\\
&\qquad = a+bk^2,
\end{aligned}$$

where p_n and q_n denote the adjuncts of P_n and Q_n respectively.

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By equating the coefficients of k^{2n} in this identity we find that, if $n > 1$,

$$\begin{aligned}\alpha Q_n q_n + \beta_0 P_n - \beta P_n p_n - \alpha_0 Q_n &= P_1 Q_{n-1}(p_1 - q_{n-1}) \\ &+ P_2 Q_{n-2}(p_2 - q_{n-2}) \\ &\cdots\cdots\cdots\cdots\cdots \\ &+ P_{n-1} Q_1 (p_{n-1} - q_1).\end{aligned}$$

If we define $P \times Q$ (adq) to denote $P \times Q(p - q)$, i.e. the continued product of P and Q and the *adjunct of the quotient* of P divided by Q, we may write the last equation in the form

$$\begin{aligned}Q_n(\alpha q_n - \alpha_0) - P_n(\beta p_n - \beta_0) &= P_{n-1} \times Q_1 (\text{adq}) + P_{n-2} \times Q_2 (\text{adq}) + P_{n-3} \times Q_3 (\text{adq}) \ldots \\ &+ P_1 \times Q_{n-1} (\text{adq}).\end{aligned}$$

The left-hand member of this equation

$$= \alpha \{Q_n (\text{ad})\} - \beta \{P_n (\text{ad})\} - \alpha_0 Q_n + \beta_0 P_n.$$

When $\beta = 1$, the equation may be written

$$\begin{aligned}&\alpha \{Q_n (\text{ad})\} - \alpha_0 Q_n + \beta_0 P_n \\ &= P_n (\text{ad}) + P_{n-1} \times Q_1 (\text{adq}) + P_{n-2} \times Q_2 (\text{adq}) \ldots + P_1 \times Q_{n-1} (\text{adq}),\end{aligned}$$

and, when $\alpha = 1$, it may be written

$$\begin{aligned}&\beta \{P_n (\text{ad})\} - \beta_0 P_n + \alpha_0 Q_n \\ &= Q_n (\text{ad}) + Q_{n-1} \times P_1 (\text{adq}) + Q_{n-2} \times P_2 (\text{adq}) \ldots + Q_1 \times P_{n-1} (\text{adq}).\end{aligned}$$

§ 43. Taking for example the first relation $K_1 E_2 - K_2 E_1 = 1$, which was written out at full length, in terms of the series, at the end of the last section, and putting for simplicity $n = 4$ (the law of the formation of the terms being as clearly seen in this particular case as in the general formula) we obtain the arithmetical theorem:

$$\begin{aligned}\frac{1^2 . 3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8^2} - \frac{1^2 . 3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8^2} (\text{ad}) &= \frac{1^2 . 3^2 . 5^2 . 7}{2^2 . 4^2 . 6^2 . 8^2} (\text{ad}) \\ &+ \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2} \times \frac{1^2}{2^2} (\text{adq}) \\ &+ \frac{1^2 . 3}{2^2 . 4^2} \times \frac{1^2 . 3^2}{2^2 . 4^2} (\text{adq}) \\ &+ \frac{1}{2^2} \times \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} (\text{adq}).\end{aligned}$$

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The left-hand member may evidently be written also in the form

$$= -\frac{1 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} \text{(ad)}.$$

Writing the adjuncts at full length, the equation is therefore

$$\begin{aligned}
&-\frac{1 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}\left(\frac{1}{1}-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\frac{2}{7}-\frac{2}{8}\right)\\
&=\frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}\left(\frac{2}{1}-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\frac{1}{7}-\frac{2}{8}\right)\\
&+\frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{1^2}{2^2}\left(\frac{2}{3}-\frac{2}{4}+\frac{1}{5}-\frac{2}{6}\right)\\
&+\frac{1^2 \cdot 3}{2^2 \cdot 4^2} \times \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\left(-\frac{1}{3}\right)\\
&+\frac{1}{2^2} \times \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\left(-\frac{1}{1}-\frac{2}{3}+\frac{2}{4}-\frac{2}{5}+\frac{2}{6}\right).
\end{aligned}$$

§ 44. Similarly from the second relation, viz. $K_1 I_2 - K_2 I_1 = 1$, we find

$$\begin{aligned}
\frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} &= \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \text{(ad)}\\
&+\frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \times \frac{1^2}{2^2} \text{(adq)}\\
&+\frac{1^2 \cdot 3}{2^2 \cdot 4} \times \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \text{(adq)}\\
&+\frac{1}{2} \times \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \text{(adq)},
\end{aligned}$$

that is, writing the adjuncts at full length,

$$\begin{aligned}
\frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} &= \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8}\left(\frac{2}{1}-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\frac{1}{7}-\frac{1}{8}\right)\\
&+\frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \times \frac{1^2}{2^2}\left(\frac{2}{3}-\frac{2}{4}+\frac{1}{5}-\frac{1}{6}\right)\\
&+\frac{1^2 \cdot 3}{2^2 \cdot 4} \times \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\left(-\frac{1}{3}+\frac{1}{4}\right)\\
&+\frac{1}{2} \times \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\left(-\frac{1}{1}+\frac{1}{2}-\frac{2}{3}+\frac{2}{4}-\frac{2}{5}+\frac{2}{6}\right).
\end{aligned}$$

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In the same way the third relation, viz. $K_1G_2 - K_2G_1 = 1$, gives

$$-\frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2} = \frac{1^2.3^2.5^2}{2^2.4^2.6^2.8}\,(\text{ad})$$

$$+\frac{1^2.3^2}{2^2.4^2.6} \times \frac{1^2}{2^2}\,(\text{adq})$$

$$+\frac{1^2}{2^2.4} \times \frac{1^2.3^2}{2^2.4^2}\,(\text{adq})$$

$$+\frac{1^0}{2} \times \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\,(\text{adq}).$$

These three equations suffice to exemplify the curious kind of arithmetical formulæ to which the 21 relations give rise. I have not examined in detail the complete system of results.

§ 45. It may be remarked that if the first term of the right-hand member of the first two of the three equations be transposed to the other side of the equation, the left-hand member becomes the same in each case, and, supposing $n = 4$ as before, we find

$$\frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2} - \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8}\,(\text{ad}) = \frac{1^2.3^2.5}{2^2.4^2.6^2} \times \frac{1^2}{2^2}\,(\text{adq})$$

$$+\frac{1^2.3}{2^2.4^2} \times \frac{1^2.3^2}{2^2.4^2}\,(\text{adq})$$

$$+\frac{1}{2^2} \times \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\,(\text{adq})$$

$$=\frac{1^2.3^2.5}{2^2.4^2.6} \times \frac{1^2}{2^2}\,(\text{adq})$$

$$+\frac{1^2.3}{2^2.4} \times \frac{1^2.3^2}{2^2.4^2}\,(\text{adq})$$

$$+\frac{1}{2} \times \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\,(\text{adq}).$$

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Expressions for the quantities E, I, G, U, V, W in terms of any one of them, § 46.

§ 46. In § 8 (p. 191) the six quantities E, I, G, U, V, W were expressed in terms of E and of I. It seems worth while to give the expressions of these quantities also in terms of G, U, V, W. The complete system of formulæ is as follows:

$$\begin{aligned}
E &= E &&= I + K,\\
I &= E - K &&= I,\\
G &= E - k'^2 K &&= I + k^2 K,\\
U &= E - \tfrac{1}{2}(1 + k'^2) K &&= I + \tfrac{1}{2} k^2 K,\\
V &= E - \tfrac{1}{2} k'^2 K &&= I + \tfrac{1}{2}(1 + k^2) K,\\
W &= E - \tfrac{1}{2} K &&= I + \tfrac{1}{2} K,
\end{aligned}$$

$$\begin{aligned}
E &= U + \tfrac{1}{2}(1 + k'^2) K &&= V + \tfrac{1}{2} k'^2 K,\\
I &= U - \tfrac{1}{2} k^2 K &&= V - \tfrac{1}{2}(1 + k^2) K,\\
G &= U + \tfrac{1}{2} k^2 K &&= V - \tfrac{1}{2} k'^2 K,\\
U &= U &&= V - \tfrac{1}{2} K,\\
V &= U + \tfrac{1}{2} K &&= V,\\
W &= U + \tfrac{1}{2} k'^2 K &&= V - \tfrac{1}{2} k^2 K,
\end{aligned}$$

$$\begin{aligned}
E &= G + k^2 K &&= W + \tfrac{1}{2} K,\\
I &= G - k^2 K &&= W - \tfrac{1}{2} K,\\
G &= G &&= W + \tfrac{1}{2}(k^2 - k'^2) K,\\
U &= G - \tfrac{1}{2} k^2 K &&= W - \tfrac{1}{2} k'^2 K,\\
V &= G + \tfrac{1}{2} k'^2 K &&= W + \tfrac{1}{2} k^2 K,\\
W &= G + \tfrac{1}{2}(k'^2 - k^2) K &&= W.
\end{aligned}$$

Change of q into $-q$, q^2 and $q^{\frac{1}{2}}$, § 47.

§ 47. The following table shows the transformations into each other of the quantities k, k', K, K', E, E', &c. produced by the change of q into $-q$, q^2 or $q^{\frac{1}{2}}$.

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TABLE I.

q	$-q$	q^2	$q^{\frac{1}{2}}$
k	$\frac{ik}{k'}$	$\frac{1-k'}{1+k'}$	$\frac{2k^{\frac{1}{2}}}{1+k}$
k'	$\frac{1}{k'}$	$\frac{2k'^{\frac{1}{2}}}{1+k'}$	$\frac{1-k}{1+k}$
K	$k'K$	$\frac{1}{2}(1+k')K$	$(1+k)K$
K'	$k'(K'-iK)$	$(1+k')K'$	$\frac{1}{2}(1+k)K'$
E	$\frac{E}{k'}$		$\frac{2V}{1+k}$
E'	$\frac{G'+iG}{k'}$	$\frac{2V'}{1+k'}$	
I	$\frac{G}{k'}$	$\frac{U}{1+k'}$	
I'	$\frac{I'+iE}{k'}$		$\frac{U'}{1+k}$
G	$\frac{I}{k'}$		
G'	$\frac{E'+iI}{k'}$		
U	$\frac{U}{k'}$		$\frac{2I}{1+k}$
U'	$\frac{W'+iW}{k'}$	$\frac{2I'}{1+k'}$	
V	$\frac{W}{k'}$	$\frac{E}{1+k'}$	
V'	$\frac{V'+iU}{k'}$		$\frac{E'}{1+k}$
W	$\frac{V}{k'}$		
W'	$\frac{U'+iV}{k'}$		

The gaps in the last two columns of this table correspond to results which involve kK, $k'K$, kK' or $k'K'$ as well as E, E', I, I', &c., as appears from the following two tables, which give the complete transformations of E, I, &c. and of E', I', &c.

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TABLE II.

q	$-q$	$\mathfrak{T}^2$	$\mathfrak{T}_1^2$
E	$\frac{E}{k'}$	$\frac{E+k'K}{1+k'}$	$\frac{2V}{1+k}$
I	$\frac{G}{k'}$	$\frac{U}{1+k'}$	$\frac{2(I-kK)}{1+k}$
G	$\frac{I}{k'}$	$\frac{E-k'K}{1+k'}$	$\frac{2(I+kK)}{1+k}$
U	$\frac{U}{k'}$	$\frac{1}{2}\frac{V+W-k'K}{1+k'}$	$\frac{2I}{1+k}$
V	$\frac{W}{k'}$	$\frac{E}{1+k'}$	$\frac{W+U+kK}{1+k}$
W	$\frac{V}{k'}$	$\frac{1}{2}\frac{V+W+k'K}{1+k'}$	$\frac{W+U-kK}{1+k}$

TABLE III.

q	$-q$	$\mathfrak{T}^2$	$\mathfrak{T}_1^2$
E'	$\frac{G'+iG}{k'}$	$\frac{2V'}{1+k'}$	$\frac{E'+kK'}{1+k}$
I'	$\frac{I'+iE}{k'}$	$\frac{2(I'-k'K')}{1+k'}$	$\frac{U'}{1+k}$
G'	$\frac{E'+iI}{k'}$	$\frac{2(I'+k'K')}{1+k'}$	$\frac{E'-kK'}{1+k}$
U'	$\frac{W'+iW}{k'}$	$\frac{2I'}{1+k'}$	$\frac{1}{2}\frac{V'+W'-kK'}{1+k}$
V'	$\frac{V'+iU}{k'}$	$\frac{W'+U'+k'K'}{1+k'}$	$\frac{E'}{1+k}$
W'	$\frac{U'+iV}{k'}$	$\frac{W'+U'-k'K'}{1+k'}$	$\frac{1}{2}\frac{V'+W'+kK'}{1+k}$

In connexion with the forms of some of the expressions in Tables II. and III. it may be remarked that

$$V + W = E + U,$$
$$W + U = I + V,$$
$$U + V = G + W.$$

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Change of q into $iq^{\frac{1}{2}}$, § 48.

§ 48. The change of q into $iq^{\frac{1}{2}}$ is equivalent to the change of q into $q^{\frac{1}{2}}$ followed by the change of q into $-q$. As shown in § 23 (p. 203), the change of q into $iq^{\frac{1}{2}}$ converts k' into $e^{-2i\theta}$ and K' into $\frac{1}{2}e^{i\theta}(K'-iK)$, where θ denotes the modular angle, given by the equation $k = \sin\theta$.

Transforming the results in the last column of Table III. by changing q into $-q$ we find that, by the change of q into $iq^{\frac{1}{2}}$,

E' becomes $e^{-i\theta}\{G' + kk'K + i(G + kk'K')\}$,

I' „ $e^{-i\theta}(W' + iW)$,

G' „ $e^{-i\theta}\{G' - kk'K + i(G - kk'K')\}$,

U' „ $\frac{1}{2}e^{-i\theta}\{G' + W' - kk'K + i(G + W - kk'K')\}$,

V' „ $e^{-i\theta}(G' + iG)$,

W' „ $\frac{1}{2}e^{-i\theta}\{G' + W' + kk'K + i(G + W + kk'K')\}$.

Since, by this change

k' becomes $e^{-2i\theta}$,

and K' „ $\frac{1}{2}e^{i\theta}(K' - iK)$,

it follows that $k'K'$ „ $\frac{1}{2}e^{-i\theta}(K' - iK)$.

The transformations of K', I', V' were given in § 24 (p. 203), and by their means the series for

$$\frac{K}{\pi}, \quad \frac{K'}{\pi}, \quad \frac{2W}{\pi}, \quad \frac{2W'}{\pi}, \quad \frac{4G}{\pi}, \quad \frac{4G'}{\pi},$$

in terms of sines and cosines of multiples of the modular angle were found in § 25 (p. 204).

The transformations of E', G', U', W' give no fresh results, except the series

$$\frac{4kk'K}{\pi} = \cos\theta - \frac{1.3}{2^2}\cos 3\theta - \frac{1^2.7}{2^2.4^2}\cos 7\theta - \frac{1^2.3^2.11}{2^2.4^2.6^2}\cos 11\theta - \&c.,$$

$$\frac{4kk'K'}{\pi} = \sin\theta + \frac{1.3}{2^2}\sin 3\theta + \frac{1^2.7}{2^2.4^2}\sin 7\theta + \frac{1^2.3^2.11}{2^2.4^2.6^2}\sin 11\theta + \&c.$$

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These formulæ may however be derived at once from the series for G by simple differentiation, since

$$\frac{dG}{d\theta} = kk'K,$$

or, even more simply, by multiplying the series for $\frac{K}{\pi}$ by $2\sin 2\theta$ and using the formula

$$2\sin 2\theta \sin(4n+1)\theta = \cos(4n-1)\theta - \cos(4n+3)\theta.$$

April 20, 1885.

Prof. FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:—

(1) *Note on the rotation of the plane of polarization of light by a moving medium.* By Prof. J. J. THOMSON, M.A.

In a paper in the *Philosophical Magazine* for April 1880, I considered, assuming the Electromagnetic Theory of Light, some of the effects produced by the motion of the medium which is the seat of the electrostatic action. The motion was then supposed to be translational. In this note I shall consider the case when the motion of the medium is of the most general character which a *rigid* body can possess.

The notation is as follows:

f, g, h are the components of the electric displacements parallel to the axes of x, y, z respectively.

a, b, c the components of magnetic induction.

F, G, H the components of the vector potential.

P, Q, R the components of the electromotive force.

p, q, r the components of the velocity of the medium; if as we shall suppose the medium moves like a rigid body, we may look on the velocity as made up of a motion of translation whose components are u, v, w and a rotatory motion the components of whose angular velocity are $\omega_1, \omega_2, \omega_3$.

μ is the magnetic permeability and K the specific inductive capacity of the medium.

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Then the equations, which give p, q, r in terms of u, v, w, ω_1, ω_2, ω_3 are

$$p = u + \omega_2 z - \omega_3 y \quad \ldots\ldots(1),$$
$$q = v + \omega_3 x - \omega_1 z \quad \ldots\ldots(2),$$
$$r = w + \omega_1 y - \omega_2 x \quad \ldots\ldots(3).$$

The equations of the electromagnetic field are

$$P = cq - br - \frac{dF}{dt} - \frac{d\psi}{dx} \quad \ldots\ldots(4),$$

$$Q = ar - cp - \frac{dG}{dt} - \frac{d\psi}{dy} \quad \ldots\ldots(5),$$

$$R = bp - aq - \frac{dH}{dt} - \frac{d\psi}{dz} \quad \ldots\ldots(6).$$

$$a = \frac{dH}{dy} - \frac{dG}{dz} \quad \ldots\ldots(7),$$

$$b = \frac{dF}{dz} - \frac{dH}{dx} \quad \ldots\ldots(8),$$

$$c = \frac{dG}{dx} - \frac{dF}{dy} \quad \ldots\ldots(9),$$

and if the only currents which exist in the medium are displacement currents, we have

$$4\pi\mu \frac{df}{dt} = \frac{dc}{dy} - \frac{db}{dz} \quad \ldots\ldots(10),$$

$$4\pi\mu \frac{dg}{dt} = \frac{da}{dz} - \frac{dc}{dx} \quad \ldots\ldots(11),$$

$$4\pi\mu \frac{dh}{dt} = \frac{db}{dx} - \frac{da}{dy} \quad \ldots\ldots(12).$$

Since $$P = \frac{4\pi}{K} f, \quad Q = \frac{4\pi}{K} g, \quad R = \frac{4\pi}{K} h,$$

we get if we differentiate (4) with respect to y, (5) with respect to z and subtract

$$\frac{4\pi}{K}\left\{\frac{df}{dy} - \frac{dg}{dx}\right\} = \left\{p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right\} c + a\omega_2 - b\omega_1 + \frac{dc}{dt} \ldots (13).$$

Similarly

$$\frac{4\pi}{K}\left\{\frac{dh}{dx} - \frac{df}{dz}\right\} = \left\{p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right\} b + c\omega_1 - a\omega_3 + \frac{db}{dt} \ldots (14),$$

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differentiating (13) with regard to y, (14) with regard to z and subtracting we get, since

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

$$\frac{4\pi}{K}\left\{\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}\right\}=4\pi\mu\left\{p\frac{d}{dx}+q\frac{d}{dy}+r\frac{d}{dz}\right\}\frac{df}{dt}$$

$$-\frac{d}{dx}\{a\omega_1+b\omega_2+c\omega_3\}+\left\{\omega_1\frac{d}{dx}+\omega_2\frac{d}{dy}+\omega_3\frac{d}{dz}\right\}a+\frac{d^2f}{dt^2},$$

or as it is more convenient to write it

$$\frac{1}{\mu K}\left\{\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}\right\}=\left\{p\frac{d}{dx}+q\frac{d}{dy}+r\frac{d}{dz}\right\}\frac{df}{dt}+\omega_3\frac{dg}{dt}-\omega_2\frac{dh}{dt}+\frac{d^2f}{dt^2},$$

with similar equations for g and h.

We may easily prove that a, b, c satisfy equations of an exactly similar type.

We shall apply these equations to a very simple case, let us suppose that the light is propagated along the axis of z and that the medium is rotating round this axis with an angular velocity ω.

In this case f and g are functions of z only so that our equations become, writing v^2 for $1/\mu K$,

$$v^2\frac{d^2f}{dz^2}=\omega_3\frac{dg}{dt}+\frac{d^2f}{dt^2},$$

$$v^2\frac{d^2g}{dz^2}=-\omega_3\frac{df}{dt}+\frac{d^2g}{dt^2}.$$

Suppose a circularly polarized ray goes along the axis of z, for which $f=\alpha\sin(nt-lz)$,

$$g=-\alpha\cos(nt-lz).$$

Substituting we get

$$v^2l^2+\omega_3 n-n^2=0.$$

If the ray had been circularly polarized in the opposite sense we should have

$$v^2l^2-\omega_3 n-n^2=0.$$

If ω_3 be small compared with n, then in the first case

$$vl=n\left(1-\tfrac{1}{2}\frac{\omega_3}{n}\right).$$

And if in the second l_1 be written for l

$$vl_1=n\left(1+\tfrac{1}{2}\frac{\omega_3}{n}\right),$$

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so that the ray polarized in the opposite sense to that of the rotation of the medium travels more slowly than the one polarized in the same sense.

The ray for which the displacements are nearly parallel to x is given by

$$f = \alpha \sin(nt - lz) + \alpha \sin(nt - l_1 z),$$

or say

$$f = 2\alpha \sin(nt - lz)$$

$$g = \alpha \cos(nt - l_1 z) - \alpha \cos(nt - lz),$$

$$g = 2\alpha \sin\left(\frac{l_1 - l}{2}\right) 3 \cos(nt - lz)$$

$$= 2\alpha \frac{(l_1 - l)}{2} z \cos(nt - lz) \quad \text{approximately}$$

$$= \alpha \cdot \frac{\omega_3}{v} z \sin(nt - lz);$$

or the plane of polarization is twisted in the same direction as the rotation of the body through an angle ω_3/v per unit length. We may state this result rather more neatly by saying that the rotation of the plane of polarization after the light has travelled any distance is equal to the angle turned through by the medium in the time taken by light to traverse that distance. To twist the plane of polarization through a third of a minute after traversing 10 metres, the medium would have to make about 500 revolutions per second. This seems just to bring it within the limit of experimental verification, as the light might be reflected backwards and forwards. We must however remember that ω_3 is the angular velocity of the ether and not of the moving piece of glass or whatever we may use for the experiment.

The amount of the rotation of the plane of polarization is independent of the wave-length, so that the theory in this form would not account for the magnetic rotation of the plane of polarization where the magnitude of the rotation varies inversely as the square of the wave-length. An application of this result seems worth noticing; according to some theories there is a rotation of the ether around the lines of magnetic force, now this will produce a rotation of the plane of polarization whatever the medium may be. Now experiments with negative results have been made to detect the rotation of the plane of polarization in air and we may conclude that if the rotation had been as much as 1′ per metre it could not have escaped detection, so that even in very powerful magnetic fields the ether cannot be rotating more than 15000 times per second.

If we go to second powers of ω/n we see that the velocity of

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propagation is $v/(1 - \frac{1}{2}\omega^2/n^2)$, so that in a rotating medium there would be dispersion but the blue rays would move more slowly than the red.

(2) *On the theory of some experiments of Fröhlich on the position of the plane of polarization of light diffracted at reflexion from a grating.* By R. T. GLAZEBROOK, M.A., F.R.S.

IN Wiedemann's *Annalen* Vol. I., Fröhlich gives an account of some experiments on the polarization of light diffracted by reflection from two glass gratings, one of which was ruled with lines at a distance of ·0506 mm. apart, the other with lines at a distance of ·00617 mm.

The incident light was polarised in a plane at 45° to the plane of diffraction and the plane of polarization observed for a series of angles of diffraction and for angles of incidence of 85°, 55° and 25°.

The position of the plane of polarization of the light reflected directly at angles varying by 5° between 85° and 25° was also observed.

It was found that for the diffracted light the direction of vibration depends on the angle of incidence, the angle of diffraction and the nature of the reflecting face, but is completely independent of the distance between the lines of the grating and of the overlapping of spectra of different orders at the same point in the field of the observing telescope; while the position of the plane of polarization of the light regularly reflected agreed fairly with that given by Fresnel's theory, except near the polarising angle where Jamin's effect, which however in glass of refractive index about 1·5 is not very marked, came in.

The only other experiments which I know of on polarization by diffraction at reflexion are some alluded to in Prof. Stokes' paper in the *Transactions of the Camb. Phil. Soc.* Vol. IX. on the dynamical theory of diffraction, but not published on account of the difficulties arising from the overlapping of the images formed by light reflected from the second surface of the grating. These were avoided to some extent by Fröhlich by using a wedge-shaped piece of glass for his grating.

Fröhlich states in his paper that his results confirm the supposition that the direction of vibration is normal to the plane of polarization, but does not give any theory.

This has been partly supplied by Réthy (Wiedemann, Vol. XI., *On the polarization of diffracted light*), who has shewn that Fröhlich's experiments may be represented by the formula

$$\tan\phi = \tan\phi_0 \cos\delta + \sec\phi_0 \sin\delta \cot\rho_0 \quad \ldots\ldots\ldots(1),$$

ϕ and ϕ_0 being the azimuths of the planes of polarization of the

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diffracted and regularly reflected light measured from the plane of diffraction, δ the angle between them, and ρ_0 a constant for any given angle of incidence.

Réthy obtains this result by taking as the solution of the equations of motion in the medium due to the disturbance at one point of the grating, the values for the displacement of any point given by

$$u = \frac{d\Phi}{dy} \qquad v = -\frac{d\Phi}{dx} \qquad w = 0,$$

where

$$\Phi = \frac{A}{r} \cos 2\pi \left\{ \frac{r}{\lambda} - \frac{t}{T} + \delta \right\},$$

r being the distance between the two points in question. Now this solution corresponds to the motion which would ensue if each element of the grating, considered as a small sphere, were made to twist backwards and forwards in periodic time T about an axis, the axis of z, the same for all elements, and it is difficult to see how this motion could arise.

Réthy also considers another possible motion given by

$$u = -\frac{d^2\Phi}{dz\,dx}, \qquad v = -\frac{d^2\Phi}{dy\,dz}, \qquad w = \frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2},$$

which would arise from the action of a periodic force acting parallel to the axis of z at each point of the grating, but according to him this solution does not lead to the equations given above. It is difficult to imagine how the state of things over the grating can be that required for the first solution, and in any case the effects produced by a train of waves passing over a given element of space differ from those produced by causing the particles of the element to perform small vibrations under the action of a periodic force. In the second case the motion is symmetrical round the direction of motion of the particles, in the first case it is not. The two are dealt with in Prof. Stokes' paper on diffraction, sections 31 and 27.

My object in the present paper is to show how the formula (1) employed by Réthy may be deduced on a certain simple assumption from Prof. Stokes' results.

The assumption is that each particle of the bounding surface is performing small oscillations parallel to some fixed direction depending on the polarization of the incident light and the angle of incidence.

Consider any point O on the bounding surface and let a line drawn through this point parallel to the direction of vibration meet a unit sphere in Z (Fig. 1). Let a ray regularly reflected from O meet the sphere in R_0 and a diffracted ray in R. Then according to Prof. Stokes the directions of vibration in the rays R_0, R, so far

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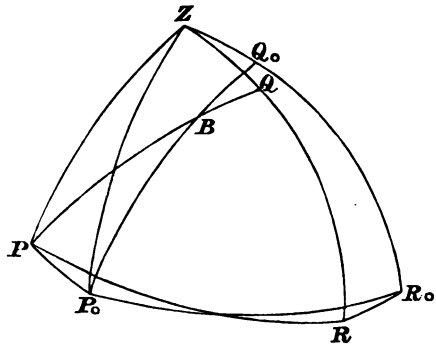
as they depend on the motion at O, lie in the planes OZR_0, OZR and are perpendicular to OR_0 and OR respectively.



Join ZR_0, ZR by great circles and take Q_0R_0, QR each equal to $\frac{1}{2}\pi$, the directions of vibration meet the sphere in Q_0 and Q.

Let P_0, P be the poles of the great circles Q_0R_0, QR respectively. Then QP, Q_0P_0, meeting in B suppose, are the traces of the wave fronts, and $PBP_0 = \delta$, the angle of diffraction. Also B is the pole of the plane of diffraction RR_0, and on Fresnel's hypothesis the azimuths of the planes of polarization of the two waves are BQ_0 and BQ respectively, while on the other hypothesis the azimuths are BP_0 and BP. Now let $ZR_0 = \rho_0$ so that ρ_0 is the angle between the direction of motion over the surface and the direction of the ray directly reflected. Then

$$ZQ_0 = \rho_0 - 90^\circ = ZP_0Q_0,$$

and since Z is the pole of PP_0, $ZP_0P = 90^\circ$ and $PP_0Q_0 = \rho_0$.

Hence the triangle PBP_0 gives

$$\sin BP_0 \cot BP = \sin PBP_0 \cot PP_0B + \cos PBP_0 \cos BP_0.$$

Or on Fresnel's hypothesis

$$\tan \phi = \cos \delta \tan \phi_0 + \sin \delta \sec \phi_0 \cot \rho_0.$$

While if we take the other hypothesis we have to put $\frac{\pi}{2} - \phi$ and $\frac{\pi}{2} - \phi_0$ for ϕ and ϕ_0 and get

$$\cot \phi = \cos \delta \cot \phi_0 + \sin \delta \operatorname{cosec} \phi_0 \cot \rho_0.$$

If we are considering diffraction by transmission, the incidence being direct, this gives Prof. Stokes' well-known law, for then, taking as the ray corresponding to ρ_0, ϕ_0, the ray transmitted directly, it is clear that the disturbance over the diffracting sur-

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face will be in that surface since the incident vibrations are parallel to the surface, and thus $\rho_0 = 90$.

Réthy has made his comparison with Fröhlich's experiments by putting the equation into the form

$$\tan \phi = a \cos \delta + b \sin \delta = c \sin (\delta + \epsilon),$$

and then calculating c and ϵ by the method of least squares. In each series there are some sixteen or eighteen observations which are sufficient for comparison with a formula containing two arbitrary constants.

For the angles of incidence 25° and 85° the same values of c and ϵ were obtained from the two gratings, for the angle 55° the values for the two were slightly different, but 55° is near the polarizing angle for the glass used and the effects of elliptic polarization observed by Jamin, Quincke and others come in.

From Réthy's values of c and ϵ I have calculated the values of ρ_0 and ϕ_0, the latter of which was independently observed for the different cases by Fröhlich.

Angle of incidence.	ρ_0	ϕ_0 calculated.	ϕ_0 observed.
Broad Grating 25	110° 58′	− 37° 34′	− 37° 1′
Broad Grating 55°	50° 48′	2° 14′	1° 52′
Broad Grating 85°	58° 7′	39° 8′	39° 32′
Narrow Grating 25°	110° 58′	− 37° 34′	− 37° 18′
Narrow Grating 55°	52° 23′	− 2° 8′	− 0° 56′
Narrow Grating 85°	58° 7′	39° 8′	39° 37′

Thus the values of ϕ_0 calculated on the assumption that Fröhlich's experiments can be represented by the equation (1) and determined directly from experiment do not differ very greatly, and the differences throughout Réthy's Tables are quantities of the same order as those shewn above.

This close agreement would seem to indicate that we may treat the motion of the ether particles on the interface, to the degree of accuracy required for these experiments, as rectilinear, while the results, as Réthy has shewn, are decisively against the hypothesis that the vibrations lie in the plane of polarization.

The calculated values of ϕ_0 given above are also given by Réthy in the course of his tables.

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May 11, 1885.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:—

(1) *On the formation of lactic acid, creatine and urea in muscular tissue.* By Prof. LATHAM, M.D.

THE variety of lactic acid which is obtained by the disintegration of albuminous compounds and is formed during the contraction or tetanus of muscular fibres, and hence called sarco-lactic acid, though agreeing in some respects with fermentation or ethidene lactic acid, differs from it in being optically active, turning the plane of polarization to the left, whereas the other is optically inactive. The difference too between their calcium and zinc salts is very marked. In rheumatic fever, again, it is formed in large quantities in the system, and differing as above stated, from fermentation lactic acid, the inference is obvious that it cannot be directly formed from the glycogen in the tissues, or from saccharine or starchy matters introduced into the digestive tract. In fact, in the disorder referred to, one of the most important aids in cutting short the malady, and putting a stop to the excessive formation of lactic acid, is to place the patient entirely on a diet of farinaceous food and milk.

Where then does it come from? The question possesses much more than a physiological interest, for if rightly answered a considerable insight will be gained into the pathology of the disorder to which I have referred.

Sarco-lactic acid may be regarded as a mixture of two kinds of lactic acid* the more abundant being paralactic acid or ethidene lactic acid $CH_3 . CH \begin{cases} OH \\ COOH \end{cases}$, the other ethene lactic acid $CH_2 . CH_2 \begin{cases} OH \\ COOH \end{cases}$. Now these two acids may be obtained,

(i) by oxidising ethylic alcohol, treating the aldehyde so obtained with hydrocyanic acid to form a cyanhydrin, and acting upon this with acids or alkalis—

$$\underset{\text{ethyl alcohol}}{C_2H_5 . HO} + O = \underset{\text{aldehyde}}{CH_3 . CHO} + H_2O$$

$$\underset{\text{aldehyde}}{CH_3 . CHO} + CNH = \underset{\text{cyanhydrin}}{CH_3 . CH \begin{cases} OH \\ CN \end{cases}}$$

$$CH_3 . CH \begin{cases} OH \\ CN \end{cases} + 2H_2O = NH_3 + \underset{\text{ethidene lactic acid}}{CH_3 . CH \begin{cases} OH \\ COOH \end{cases}} \dagger$$

* Watts, *Dictionary of Chemistry*, Vol. VIII. p. 1160.
† Fownes, *Manual of Chemistry*, 1877, p. 625.

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(ii) By converting ethene alcohol or glycol into a cyanhydrin and boiling with acids or alkalis—

$$\begin{matrix}CH_2.OH\\|\\CH_2.OH\\\text{glycol}\end{matrix} + HCl = H_2O + CH_2\begin{cases}CH_2OH\\Cl\end{cases}$$

chlorhydrin

$$CH_2\begin{cases}CH_2.OH\\Cl\end{cases} + KCN = KCl + CH_2\begin{cases}CH_2OH\\CN\end{cases}$$

cyanhydrin

$$CH_2\begin{cases}CH_2OH\\CN\end{cases} + 2H_2O = CH_2\begin{cases}CH_2OH\\COOH\end{cases} + NH_3*$$

ethene lactic acid

In a paper printed in the *Cambridge Philosophical Proceedings*, in 1882, I endeavoured to show that albumen might be regarded as a compound of cyanhydrins or cyan-alcohols, bodies having the composition generally of $R...CH\begin{cases}HO\\CN\end{cases}$; and I showed how by combining these cyan-alcohols together in certain proportions a compound might be obtained having very nearly the same composition as that given by Schützenberger for albumen. The combination of molecules of the form $R...CH\begin{cases}OH\\CN\end{cases}$ which I suggested gave with H_2SO_3 the compound

$$\left(C_7H_6\begin{cases}OH\\CN\end{cases}\right)\left(C_5H_{10}\begin{cases}OH\\CN\end{cases}\right)_6\left(CH_2\begin{cases}OH\\CN\end{cases}\right)_9\left(C_3H_6\begin{cases}OH\\CN\end{cases}\right)_2\left(C_3H_5\begin{cases}OH\\OH\\CN\end{cases}\right)\left(C_2H_4\begin{cases}OH\\CN\end{cases}\right)_2H_2SO_3$$

$$= C_{80}H_{133}N_{21}O_{25}S$$

three molecules of this undergoing condensation giving as the composition of albumen

$$\underset{\text{albumen}}{C_{240}H_{399}N_{63}O_{75}S_3}$$

which differs from Schützenberger's formula, $C_{240}H_{387}N_{65}O_{75}S_3$†, only in the small amounts of hydrogen and nitrogen.

Now I did not suggest that this was anything more than an approximation to the constitution of albumen, in fact I would modify it materially now, as I shall show directly; but the more the subject is considered the more evident does it become that the fundamental point with which I started (viz. that the cyan-alcohols are the bodies from which this complex molecule is built up)—is the key, not only to the constitution of that body but to the changes which it undergoes both in the system and after death.

* Fownes, *Manual of Chemistry*, 1877, p. 319.
† *Annales de Chimie*, 1879, p. 384.

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The chief objection to the formula which I have suggested for albumen is "that it consists of a large number of molecules in *molecular* as distinguished from *atomic* combination; and on the whole, organic molecules seem to have extremely little tendency to enter into this kind of combination." This objection is fatal to my formula, but stimulated by the kind expressions of the writer of this criticism I did not relinquish the subject, and I hope to show now that a further step has been gained towards the solution of the problem.

When proteid substances are decomposed or when they are introduced into the alimentary canal and digested, certain amido-acids, or glycines, such as glycocine $CH_2\begin{cases}NH_2\\COOH\end{cases}$, leucine $C_5H_{10}\begin{cases}NH_2\\COOH\end{cases}$, tyrosine $C_6H_4\begin{cases}OH\\C_2H_3(NH_2)COOH\end{cases}$ &c., are produced, these substances are absorbed into the system and thereby the waste of the tissues is repaired. Now these glycines are capable of uniting with each other (Hofmeister) and it is probable that their molecular weights are at least double as great as their formulæ would indicate*. They may be represented by the formula

$$\begin{array}{l} C_nH_{2n}.NH_2.O.CO \\ \quad\quad\quad\

Similarly

$$3CH_2\begin{cases}NH_2\\COOH\end{cases}\quad\text{glycocine} \qquad = \begin{matrix}CH_2\begin{cases}NH_2\\CO.NH\end{cases}\\ CH_2\begin{cases}CO.NH\\COOH\end{cases}\end{matrix} + 2H_2O$$

Similarly

$$3C_2H_4\begin{cases}NH_2\\COOH\end{cases}\quad\text{alanine} \qquad = \begin{matrix}C_2H_4\begin{cases}NH_2\\CONH\end{cases}\\ C_2H_4\begin{cases}CO.NH\\COOH\end{cases}\end{matrix} + 2H_2O$$

$$\text{and } 3C_5H_{10}\begin{cases}NH_2\\COOH\end{cases}\quad\text{leucine} \qquad = \begin{matrix}C_5H_{10}\begin{cases}NH_2\\CO.NH\end{cases}\\ C_5H_{10}\begin{cases}CO.NH\\COOH\end{cases}\end{matrix} + 2H_2O$$

and in the same way these compound molecules may be connected together by combining the COOH in one with the NH_2 in the other, with elimination of H_2O.

If these are the changes which take place on the dehydration of these bodies in their progress to the formation of albumen, a very interesting point attracts attention, viz. that the quantivalence of the atoms in the above molecules of CO . NH is not satisfied = C = O . = N – H, there is a break between the oxygen and nitrogen; there must therefore be an interchange of atoms and the molecule becomes = C = N – O – H or CN . OH. Such a change coincides with the view expressed by Pflüger* that ammonium cyanate $CNO.NH_4$ is a type of living, and urea $CO\begin{cases}NH_2\\NH_2\end{cases}$ of dead nitrogen, and the conversion of the former into the latter is an image of the essential change which takes place when a proteid dies.

By dehydration of the glycines then we should have generally:

$$3C_nH_{2n}\begin{cases}NH_2\\COOH\end{cases} \qquad = \begin{matrix}C_nH_{2n}\begin{cases}NH_2\\CN.OH\end{cases}\\ C_nH_{2n}\begin{cases}CN.OH\\COOH\end{cases}\end{matrix} + 2H_2O$$

* Archiv, Bd. x. s. 337.

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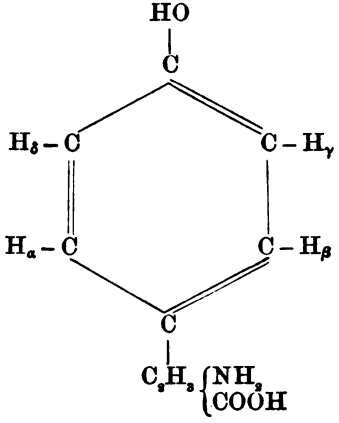
Similarly

$$6C_nH_{2n}\begin{cases}NH_2\\COOH\end{cases} = (4)\ C_nH_{2n}\begin{cases}NH_2\\CN.OH\\CN.OH\\COOH\end{cases} + 5H_2O$$

writing it so for shortness, the number in the bracket indicating the number of molecules of $C_nH_{2n}\begin{cases}OH\\CN\end{cases}$; and in this way we arrive at an *atomic* combination of the cyanhydrins or cyan-alcohols united to a cyanamide $C_nH_{2n}\begin{cases}NH_2\\CN\end{cases}$ and an acid $C_nH_{2n}\begin{cases}OH\\COOH\end{cases}$.

If now we dehydrate six molecules of glycocine and three of each of the other glycines in the series up to leucine and attach them to a benzene nucleus, we shall I think have a compound the constitution of which will give us some insight into that of albumen.

Tyrosine $C_6H_4\begin{cases}OH\\C_2H_3(NH_2)COOH\end{cases}$ or para-oxyphenyl-amido propionic acid, which is obtained from proteid substances, may be thus represented



The $C_2H_3\begin{cases}NH_2\\COOH\end{cases}$ being derived according to my view from $C_2H_4\begin{cases}NH_2\\CN\end{cases}$ that is from $C_2H_4\begin{cases}OH\\CN\end{cases}$. This indicates the point of attachment to the benzene nucleus of the C_2H_4 series. If then we replace H_α by the CH_2 series, H_β by the C_3H_6 and C_4H_8 series, and H_γ by the C_5H_{10} series, we have H_δ left to be replaced by some sulphur compound and the composition of Taurine $C_2H_4\begin{cases}NH_2\\SO_3H\end{cases}$

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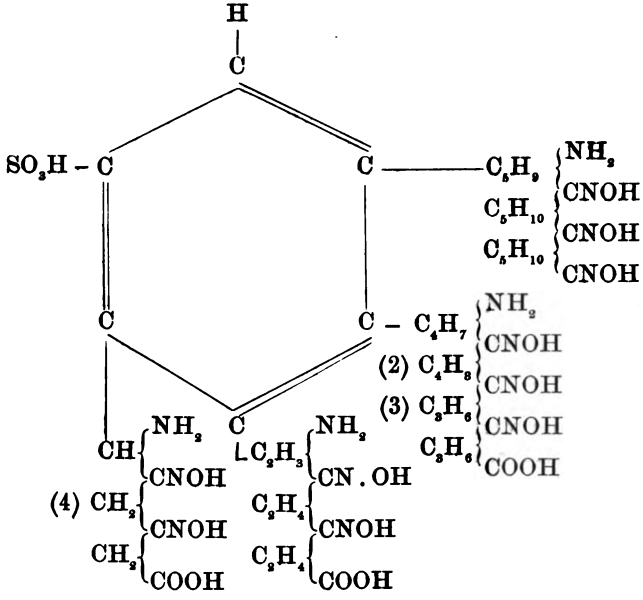
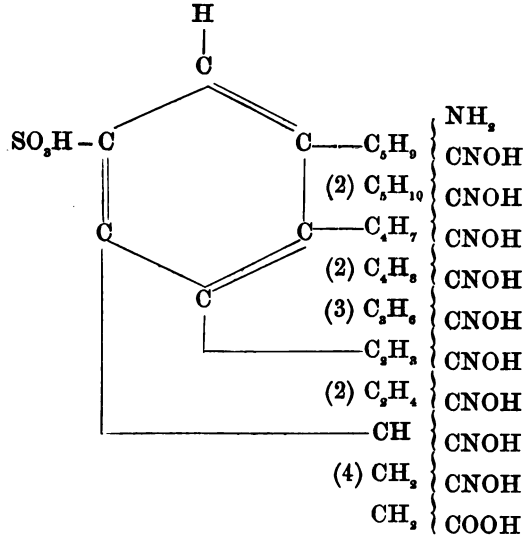
suggests SO_3H. The compound would then be (OH being replaced by H);



or if we combine the series together, the molecule COOH in the one combining with the NH_2 of the other with elimination of H_2O, the compound may be represented by the formula:



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the numbers in the brackets representing the number of times that cyanhydrin is repeated in that group, and the composition of the compound would be $C_{72}H_{113}N_{18}O_{22}S$; which is almost identical with the formula given by Lieberkühn viz. $C_{72}H_{112}N_{18}O_{22}S$.

I offer this then as showing how by combining the cyan-alcohols with a benzene nucleus a body having the same composition as albumen may be built up:—not as asserting that it is the actual composition, though strongly impressed that this is the case. For by simply dehydrating six molecules each of glycocine, alanine and leucine and combining them in a similar fashion with a benzene nucleus, we should arrive at a formula $C_{72}H_{115}N_{18}O_{22}S$; but as I shall show presently there are reasons for believing that the molecules $C_3H_6\begin{cases}OH\\CN\end{cases}C_4H_8\begin{cases}OH\\CN\end{cases}$ are constituents of albumen.

Having arrived at the constitution of albuminous matter, we are now prepared to consider the changes which result from its disintegration.

From the various cyan-alcohols when treated with acids or alkalis the various acids of the lactic series can be obtained

$$C_nH_{2n}\begin{cases}OH\\CN\end{cases}+2H_2O=C_nH_{2n}\begin{cases}OH\\COOH\end{cases}+NH_3$$

lactic, oxybutyric &c. are examples.

The various glycines or amido-acids can also be obtained from them by the following methods. By digesting the cyan-alcohols with ammonia the cyanamides are obtained*

$$C_nH_{2n}\begin{cases}OH\\CN\end{cases}+NH_3=H_2O+C_nH_{2n}\begin{cases}NH_2\\CN\end{cases}$$

* *Berichte der deutsch. chem. Gesell.* XIV. s. 1985. "The amido acids of the fatty series are easily obtained by the familiar reactions which take place on treating aldehyde ammonia with hydrochloric and hydrocyanic acids, and which led Strecker to the discovery of alanine. . . . . The reactions indicated by Strecker take place unquestionably according to the following general formulæ:

$$R\ldots C\begin{cases}NH_2\\H\\OH\end{cases}+HCN=R\ldots C\begin{cases}NH_2\\H\\CN\end{cases}+H_2O,$$

and $$R\ldots C\begin{cases}NH_2\\H\\CN\end{cases}+2H_2O+HCl=R\ldots CH(NH_2)\ldots COOH+H_4NCl.$$

The question arises, whether the cyanamide

$$R\ldots C\begin{cases}NH_2\\H\\CN\end{cases}$$

could not be obtained more readily from the cyanhydrides of the aldehydes

$$R\ldots C\begin{cases}CN\\H\\OH\end{cases}$$

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which acted upon by acids or alkalis produce the glycines or amido-acids, glycocine, alanine, leucine, &c.

$$C_nH_{2n}\begin{cases}NH_2\\CN\end{cases} + 2H_2O = C_nH_{2n}\begin{cases}NH_2\\COOH\end{cases} + NH_3.$$

If now muscular tissue is such a compound as I have suggested, the molecules being held together by some force, vital or otherwise, it is not difficult to imagine that in the normal disintegration of the tissue some such changes as the following may take place.

First the molecule $CH_2\begin{cases}OH\\COOH\end{cases}$ would be detached and when oxidised would be converted into methyl-aldehyde, CO_2 and water,

$$CH_2\begin{cases}OH\\COOH\end{cases} + O = H.CHO + CO_2 + H_2O$$

glycollic acid methyl aldehyde

the aldehyde being then either combined with a fresh molecule of HCN to form a cyan-alcohol, or further oxidised into CO_2 and H_2O. We should then have attached to the benzene nucleus a compound of cyan-alcohols by the hydration or decomposition of one molecule of which we should have the corresponding acid formed and ammonia liberated; the nascent ammonia may however combine to form a cyanamide with the next molecule in the chain, which hydrated would form the amido-acid, ammonia being again liberated and forming a cyanamide with the next molecule higher up; and so on all through the different series. The changes then may be thus represented

$$\begin{matrix}C_2H_4\begin{cases}OH\\CN.OH\end{cases}\\CH_2\begin{cases}CN.OH\\CN.OH\end{cases}\\CH_2\begin{cases}CN.OH\\CN\end{cases}\end{matrix} + H_2O = \begin{matrix}C_2H_4\begin{cases}OH\\CN.OH\end{cases}\\CH_2\begin{cases}CN.NH_2\\CN\end{cases}\end{matrix} + CH_2\begin{cases}OH\\COOH\end{cases}$$

glycollic acid.

by digesting them with ammonia, expecting the ultimate change to be as follows:

$$R...C\begin{cases}CN\\H\\OH\end{cases} + NH_3 = R...C\begin{cases}CN\\H\\NH_2\end{cases} + H_2O$$

The truth of this supposition has been confirmed by experiment."—*Berichte*, XIII. s. 382.

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Three different conditions now present themselves; (i) the molecule $CH_2\begin{cases}NH_2\\CN\end{cases}$ may become detached, or (ii) the molecule $CN\,.\,NH_2$ or (iii) the larger molecule $\begin{matrix}CH_2\\CH_2\end{matrix}\begin{cases}OH\\CN\,.\,NH_2\\CN\end{cases}$ may be separated from the chain. Let us consider the results in cases (i) and (ii) taking the consideration of (iii) further on. In case (i) we should have by hydration

$$\begin{matrix}C_2H_4\\CH_2\\CH_2\end{matrix}\begin{cases}OH\\CN\,.\,OH\\CN\,.\,NH_2\\CN\end{cases} \qquad \begin{matrix}C_2H_4\\CH_2\end{matrix}\begin{cases}OH\\CN\,.\,NH_2\\CN\end{cases} + CH_2\begin{cases}NH_2\\COOH\end{cases}$$

$$+\,H_2O =$$

and so on, the remaining two molecules by hydration being converted into $CH_2\begin{cases}NH_2\\COOH\end{cases}$ glycocine, $C_2H_4\begin{cases}NH_2\\COOH\end{cases}$ alanine, and NH_3, which last passes on to form another cyanamide in the chain.

If on the other hand the molecule $CN\,.\,NH_2$ is detached we have

$$\begin{matrix}C_2H_4\\CH_2\\CH_2\end{matrix}\begin{cases}OH\\CN\,.\,OH\\CN\,.\,NH_2\\CN\end{cases} = \begin{matrix}C_2H_4\\CH_2\\CH_2\end{matrix}\begin{cases}OH\\CN\,.\,OH\\\\CN\end{cases} + CN\,.\,NH_2$$

and there is formed the next cyan-alcohol in the series

$$CH_2\,.\,CH_2\begin{cases}OH\\CN\end{cases} \text{ or } C_2H_4\begin{cases}OH\\CN\end{cases},$$

and cyanamide; the latter with water forming urea*

$$CN\,.\,NH_2 + H_2O = CO\begin{cases}NH_2\\NH_2\end{cases}$$

urea.

* Fownes' *Chemistry*, p. 406.

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In this way we pass from the lower cyan-alcohols to the higher with the formation of urea, the two molecules of CH_2 combining to form C_2H_4. Similarly

$$C_2H_4\begin{cases}OH\\CN.NH_2\\CN\end{cases} = C_2H_4,\ CH_2\begin{cases}OH\\CN\end{cases} + CN.NH_2$$

$$= C_3H_6\begin{cases}OH\\CN\end{cases} + CN.NH_2$$

and

$$C_3H_6\begin{cases}OH\\CN.NH_2\\C_4H_4\begin{cases}CN\end{cases}\end{cases} = C_5H_{10}\begin{cases}OH\\CN\end{cases} + CN.NH_2$$

Here we have not only an explanation of the formation of urea in the tissues but the reason why the amido-bodies obtained from the tissues possess different properties from those made in the laboratory. It may easily be shown that the above cyan-alcohol $C_5H_{10}\begin{cases}OH\\CN\end{cases}$, from which leucine may be prepared, will contain six different forms of $C_5H_{10}\begin{cases}OH\\CN\end{cases}$.

Going back now; if urea and the next higher cyan-alcohol in the series are formed, this latter by hydration may be converted into the corresponding acid and ammonia

$$CH_2.CH_2\begin{cases}OH\\CN\end{cases} + 2H_2O = CH_2.CH_2\begin{cases}OH\\COOH\end{cases} + NH_3$$

Lactic acid

which last again passes on to form a cyanamide with another molecule in the chain.

We may conceive that under the control of nervous force, varying in intensity, these changes may take place and that glycocine, alanine, amido-butyric and amido-valeric acids and leucine result, bodies which we know can be obtained from muscular tissue. Further, that the cyan-alcohols of one series may be as I have shown transformed into those of a higher series, and that that particular cyan-alcohol may be converted into the corresponding acid; lactic acid, for example, being formed from glycocine by its dehydration and conversion into $CH_2\begin{cases}OH\\CN\end{cases}$, and subsequent hydration and condensation into urea and $C_2H_4\begin{cases}OH\\CN\end{cases}$.

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By oxidation, lactic acid and the others belonging to the series may be converted into the corresponding aldehyde, carbonic acid and water.

$$C_2H_4\begin{cases}OH\\COOH\end{cases} + O = CH_3 . CHO + CO_2 + H_2O$$

lactic acid — aldehyde

or generally

$$C_nH_{2n}\begin{cases}OH\\COOH\end{cases} + O = C_{n-1}H_{2n-1} . CHO + CO_2 + H_2O$$

acid — aldehyde

The aldehyde may then be further oxidised in the system into carbonic acid and water

$$C_{n-1}H_{2n-1} . CHO + O_{3n} = nCO_2 + nH_2O,$$

or combining with newly formed HCN may produce a fresh cyan-alcohol and so again take its place in the tissue.

Or the aldehyde from lactic acid may combine with the SO_3H disengaged from the benzene nucleus forming $C_2H_4\begin{cases}OH\\SO_3H\end{cases}$ which combining with ammonia produces $C_2H_4\begin{cases}OH\\SO_3NH_4\end{cases}$, and this in the laboratory may be transformed into Taurine $C_2H_4\begin{cases}NH_2\\SO_3H\end{cases}$. A comparison of the graphic formula of Tyrosine with that of albumen already given shows how this body is derived.

Suppose now the bond connecting the benzene nucleus was broken at the point connecting it with the cyan-alcohol $C_2H_4\begin{cases}OH\\CN\end{cases}$ and separated from the CH_2 series; the latter would then undergo the changes already described, whereas the former would be converted into lactic acid and ammonia, the latter combining with the next molecule of $C_2H_4\begin{cases}OH\\CN\end{cases}$ to form a cyanamide. In this way lactic acid would be formed at two different points, and being derived from ethene and ethidene cyanhydrins would possess the characters of sarco-lactic acid.

Let us go back and consider case (iii) where the molecule $CH_2\begin{cases}OH\\CN\end{cases} . NH_2$ is detached from the chain; if after its separation $CH_2\begin{cases}\\CN\end{cases}$

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$CN.NH_2$ is detached, and takes the form $C\begin{cases}NH\\NH\end{cases}$ we should have

$$CH_2\begin{cases}OH\\CN.NH_2\end{cases} = C\begin{cases}NH\\NH\end{cases} + \begin{matrix}CH_2-OH\\|\\CH_2-CN\end{matrix}$$

$$CH_2\begin{cases}\\CN\end{cases}$$

$$= HN=C\begin{cases}NH.CH_2-CH_2-CN.OH\end{cases}$$

$$= HN=C\begin{cases}N.CH_2-CH_2-CN.OH\end{cases}$$

But as I have previously pointed out $CN.OH$ represents living, $CO.NH$ dead nitrogen; the formula therefore becomes

$$= HN=C\begin{cases}N.CH_2-CH_2-CO.NH-\end{cases}$$

$$= NH=C\begin{cases}NH- \quad CO\\N.CH_2-CH_2\end{cases}$$

The ordinary formula for creatinine*.

The change is perhaps more intelligible if we consider the formation of creatine. If before the separation of the molecule $CN.NH_2$ we hydrate the compound, we have

$$CH_2\begin{cases}OH\\CN.NH_2\end{cases} + H_2O = CH_2\begin{cases}NH_2\\CN.NH_2\\COOH\end{cases}$$

$$= CN.NH_2 + \begin{matrix}CH_2-NH_2\\|\\CH_2-COOH\end{matrix}$$

$$= C\begin{cases}HN\\HN\end{cases} + \begin{matrix}CH_2-NH_2\\|\\CH_2-COOH\end{matrix}$$

$$= NH=C\begin{cases}NH_2\\NH.CH_2-CH_2-COOH\end{cases}$$

$$= NH=C\begin{cases}NH_2\\N.CH_2-CH_2-COOH\end{cases}$$

* Fownes, p. 614.

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The ordinary formula for creatine—and according to this we ought to obtain creatine artificially from ethene alanine.

I have thus endeavoured to show that albumen is a compound of cyan-alcohols united to a benzene nucleus, these being derived, as I have pointed out in my previous paper, from the various aldehydes, glycols and ketones; that lactic acid is obtained in two ways, either from $C_2H_4\begin{cases}OH\\ON\end{cases}$, or from changes and condensation in $CH_2\begin{cases}OH\\CN\end{cases}$, from which latter creatine and creatinine are also derived; and that urea may be obtained from one series of cyan-alcohols with the production of a cyan-alcohol higher in the series.

There are still one or two other points that I do not wish to leave unnoticed. According to Tieman, the cyanamides $R\,C\begin{cases}NH_2\\H\\CN\end{cases}$ are very unstable bodies and with the elimination of NH_3 very easily condense into Imido-nitriles*.

$$2\{R-CH(NH_2)\ldots CN\} = \begin{matrix}R-CH-CN\\ \diagdown\\ \quad NH\\ \diagup\\ R-CH-CN\end{matrix} + NH_3$$

and

$$\begin{matrix}R-CH-CN\\ \diagdown\\ \quad NH\\ \diagup\\ R-CH-CN\end{matrix} + \{R-CH(NH_2)-CN\}$$

$$= NH_3 + \begin{matrix}R-CH-CN.R\\ \diagdown\\ \quad N-CH\\ \diagup\quad\quad\diagdown\\ R-CH-CN\quad\quad CN\end{matrix}$$

If then the force holding the cyan-alcohols composing living proteid together were suddenly withdrawn, changes would quickly take place in these unstable bodies; there would be the formation of some acid and the different cyanamides, which latter would undergo the condensation above described. Does this not offer some clue to the phenomena of rigor mortis and the coagulation of the blood?

* *Berichte*, xiv. s. 1958.

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Again, by combining two molecules of $CH_2\begin{cases}NH_2\\CN\end{cases}$ we should have by Tieman's formula

$$2CH_2\begin{cases}NH_2\\CN\end{cases} = \begin{matrix}CH_2-CN\\ \diagup\\ NH\\ \diagdown\\ CH_2-CN\end{matrix} + NH_3$$

which when hydrated with weak acids would give

$$C_2H_4(NH)\begin{cases}COOH\\COOH\end{cases}$$

a body having the same composition as aspartic acid, but differing in that it is an *imido* instead of an *amido* body. Similarly

$$CH_2\begin{cases}NH_2\\CN\end{cases} + C_2H_4\begin{cases}NH_2\\CN\end{cases} = \begin{matrix}C_2H_4-CN\\ \diagup\\ NH\\ \diagdown\\ CH_2-CN\end{matrix} + NH_3$$

which hydrated would give $C_3H_6(NH)\begin{cases}COOH\\COOH\end{cases}$ a body having the same composition as glutamic acid, differing only in structure. But by strong HCl and high temperature these 'imido-nitriles' seem always to give aldehyde, HCN and an amido acid. It is not improbable therefore that under certain conditions we may have

$$2CH_2\begin{cases}NH_2\\CN\end{cases} \text{ converted into } \underset{\text{aspartic acid}}{CH_2.CH(NH_2)\begin{cases}COOH\\COOH\end{cases}}$$

and $CH_2\begin{cases}NH_2\\CN\end{cases} + C_2H_4\begin{cases}NH_2\\CN\end{cases}$ converted into $\underset{\text{glutamic acid.}}{C_3H_5.(NH_2)\begin{cases}COOH\\COOH\end{cases}}$

Lastly, a few words as to the formation of glucose. In plants CO_2 entering by the leaves combines with H_2O sent up from the roots, and from these starch is said to be formed, a volume of oxygen equal to that of the CO_2 absorbed being exhaled by the plant.

MM. Loew and Bokorny, and Pringsheim have shown that there is a substance in living plasma which has the property of reducing silver salts and for this reason is regarded as an aldehyde. The aldehyde resulting from the combination in the plant of CO_2 and H_2O. Pringsheim gives the following as the reaction; the volume of CO_2 absorbed and of O given out, being the same;

$$CO_2 + H_2O = \underset{\text{methyl aldehyde}}{H.CHO} + O_2$$

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By condensation of the aldehyde glucose is formed

$$\underset{\text{aldehyde}}{6H.CHO} = \underset{\text{glucose}}{C_6H_{12}O_6}$$

which by dehydration is converted into starch

$$\underset{\text{glucose}}{C_6H_{12}O_6} = \underset{\text{starch}}{C_6H_{10}O_5} + H_2O$$

In muscular tissue now, by hydration of $CH_2\begin{cases}OH\\CN\end{cases}$ we form glycollic acid, which when oxidised in the tissue is converted into methyl aldehyde, and this by condensation produces glucose, which by dehydration gives rise to glycogen;

$$\underset{\substack{\text{methene}\\\text{cyan-alcohol}}}{CH_2\begin{cases}OH\\CN\end{cases}} + 2H_2O = \underset{\text{glycollic acid}}{CH_2\begin{matrix}OH\\COOH\end{matrix}} = NH_3$$

$$\underset{\text{glycollic acid}}{CH_2\begin{cases}OH\\COOH\end{cases}} + O = \underset{\substack{\text{methyl}\\\text{aldehyde}}}{H.CHO} + CO_2 + H_2O$$

$$\underset{\text{aldehyde}}{6H.CHO} = \underset{\text{glucose}}{C_6H_{12}O_6}$$

and

$$\underset{\text{glucose}}{C_6H_{22}O_6} = \underset{\text{glycogen.}}{C_6H_{10}O_5} + H_2O$$

From what I have said as to the formation of urea in muscular tissue it is easy to understand why, when leucine or glycocine are introduced into the alimentary canal, a proportionate quantity of urea appears in the urine*. Taking glycocine for example, two molecules in passing from the alimentary canal to the liver are dehydrated

$$2CH_2\begin{cases}NH_2\\COOH\end{cases} = \begin{matrix}CH_2\begin{cases}NH_2\\CN.OH\end{cases}\\CH_2\begin{cases}\\COOH\end{cases}\end{matrix}$$

This now is attached to the last molecule in the albuminous chain which is undergoing change $CH_2\begin{cases}OH\\CN\end{cases}$ and we have

$$\begin{matrix}CH_2\begin{cases}NH_2\\CN.OH\end{cases}\\CH_2\begin{cases}\\COOH\end{cases}\end{matrix} + \begin{matrix}\&c.\begin{cases}\\CN.OH\end{cases}\\CH_2\begin{cases}\\CN\end{cases}\end{matrix}$$

* Foster's *Physiology*, 4th ed., p. 439.

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$$= \begin{matrix} \text{\&c.} \\ CH_2 \\ CH_2 \\ CH_2 \end{matrix} \begin{cases} \\ CN.OH \\ CN.NH_2 \\ CN.OH \\ COOH \end{cases}$$

The $CN.NH_2$ is now liberated to form urea and we have

$$\begin{matrix} \text{\&c.} \\ CH_2CH_2 \\ CH_2 \end{matrix} \begin{cases} \\ CN.OH \\ CN.OH \\ COOH \end{cases}$$

remaining, the $CH_2 \begin{cases} OH \\ COOH \end{cases}$ being converted into glycollic acid, and the $CH_2.CH_2 \begin{cases} OH \\ CN \end{cases}$ being either converted into lactic acid, and undergoing the other changes previously referred to, or being combined with NH_3 to form a cyanamide and then an amido body.

The application of the theory I have here advanced to the pathology of diabetes, of gout and of rheumatism is, I think, very evident. But this, and the conditions which lead to the defective metabolism associated with these disorders, will form the subjects of discourse in another place next month.

POSTSCRIPT.

The question may reasonably be asked;—What becomes of the glucose in the system, if lactic acid is not formed from it? The most natural answer seems to be,—that one change at least is into alcohol and carbonic acid—

$$\underset{\text{glucose}}{C_6H_{12}O_6} = 2CO_2 + 2\underset{\text{alcohol.}}{C_2H_5.HO}$$

Startling as this view at first sight appears, the following points nevertheless indicate that such a change is possible.

Oxidised in the tissues alcohol may be converted into aldehyde

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and this combining with HCN would form ethidene cyanhydrin, a constituent, as I have tried to show, of albumen

$$\underset{\text{alcohol}}{C_2H_5 . HO} + O = \underset{\text{aldehyde}}{CH_3 . CHO} + H_2O$$

$$\underset{\text{aldehyde}}{CH_3 . CHO} + HCN = \underset{\text{ethidene cyanhydrin}}{CH_3 . CH\begin{cases}OH\\CN\end{cases}}$$

this may then be decomposed into lactic acid or undergo the other changes in the tissues which I have referred to.

But there is a further purpose it may serve. On looking at the formula I have given for albumen, we see that there is still another way in which the molecules can separate in this unstable compound;—the molecules of CNOH may become detached, and, by the combination of the C_nH_{2n} portion, a cyanhydrin higher in the series will be formed. Now when CNOH is passed into alcohol the following changes take place

$$\underset{\text{cyanic acid}}{CNOH} + \underset{\text{alcohol}}{C_2H_5 . HO} = \underset{\text{urethane}}{CO\begin{cases}NH_2\\OC_2H_5\end{cases}}$$

urethane or ethylic carbamate is formed, and this combined with NH_3 will give urea*.

$$\underset{\text{urethane}}{CO\begin{cases}NH_2\\OC_2H_5\end{cases}} + NH_3 = \underset{\text{urea}}{CO\begin{cases}NH_2\\NH_2\end{cases}} + \underset{\text{alcohol.}}{C_2H_5 . HO}$$

But another substance is also formed—allophanic ether

$$2CNOH + C_2H_5 . HO = \underset{\text{allophanic ether.}}{\begin{matrix}CO\begin{cases}NH_2\\NH\end{cases}\\CO\begin{cases}NH\\OC_2H_5\end{cases}\end{matrix}}$$

This by alkalis may be transformed into a salt of the base and alcohol.

$$\underset{\text{allophanic ether}}{\begin{matrix}CO\begin{cases}NH_2\\NH\end{cases}\\CO\begin{cases}NH\\OC_2H_5\end{cases}\end{matrix}} + \underset{\text{soda}}{NaHO} = \underset{\text{alcohol}}{C_2H_5 . HO} + \underset{\text{sodium allophanate}}{\begin{matrix}CO\begin{cases}NH_2\\NH\end{cases}\\CO\begin{cases}NH\\ONa.\end{cases}\end{matrix}}$$

* Fownes, *Organic Chemistry*, 1877, p. 390.

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which is readily decomposed by weak acids into carbonic acid and urea*

$$2\,CO\begin{cases}NH_2\\NH\\ONa\end{cases} + H_2O = Na_2CO_3 + CO_2 + 2CO\begin{cases}NH_2\\NH_2\end{cases}$$

If the allophanic ether however were combined with g'ycocine we should have

$$CO\begin{cases}NH_2\\NH\\OC_2H_5\end{cases} + CH_2\begin{cases}NH_2\\COOH\end{cases} = C_2H_5.HO + CO\begin{cases}NH_2\\NH\\NH-CH_2-COOH\end{cases}$$

and this compound united with urea may, as I have elsewhere shewn†, form ammonium urate and water.

$$CO\begin{cases}NH_2\\NH\\NH-CH_2-COOH\end{cases} + CO\begin{cases}NH_2\\NH_2\end{cases} = \underset{\text{ammonium urate.}}{C_5N_4O_3H_3.NH_4} + 2H_2O$$

urea

From what I have stated it is easy to see how alcohol may act as a food, and that after its administration the amount of CO_2 or of urea eliminated from the system may not exceed the normal quantity; if anything this theory would show that the CO_2 should be lessened. It is not more difficult to conceive, *à priori*, that glucose should in the system be transformed into carbonic acid and alcohol, than that it should be converted into lactic acid; both changes are simply the result, out of the body, of different forms of so-called fermentation.

A second question may be asked. Whence is the HCN derived, that is to form the cyan-alcohols entering into the composition of albumen?

It is from the oxidation of the amido-bodies.

By oxidising glycocine for instance, we get‡

$$CH_2\begin{cases}NH_2\\COOH\end{cases} + O_2 = HCN + CO_2 + 2H_2O$$

Leucine gives§

$$C_5H_{10}\begin{cases}NH_2\\COOH\end{cases} + O_2 = C_4H_9.CN + CO_2 + 2H_2O$$

* Watts' *Dict. of Chem.*, Vol. I. p. 133.

† *On the Formation of Uric Acid in Animals*, 1884.

‡ Watts' *Dict.*, Vol. II. p. 903. § *Ib.* Vol. III. p. 582.

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By oxidising asparagin, we get*

$$C_2H_3(NH_2)\begin{cases}CONH_2\\COOH\end{cases} + O_4 = HCN + CHO_2.NH_4 + 2CO_2 + H_2O$$

asparagin — ammonium formate.

Taking this view then of the constitution of albumen, the following may be given as a summary of the nutritive changes.

The amido-acids glycocine, leucine, &c. in passing from the alimentary canal to the liver are dehydrated, forming a series of cyanhydrins or cyan-alcohols grouped around a benzene nucleus, and then pass into the circulation. In the circulation there is constant change going on, and the cyan-alcohols hydrated are converted into the glycines or amido-acids, which as they pass into the tissues are again transformed into the cyan-alcohols and there partly hydrated and then oxidised give rise to the various effete products which are eliminated from the system chiefly in the form of carbonic acid and urea.

(2) *On the molecular theory of viscous solids.* By Rev. C. TROTTER, M.A.

THE following paper is an attempt to give somewhat greater definiteness to our ideas of what may be called the molecular architecture of a viscous body.

I assume with most modern physicists that the structure of all bodies is molecular, i.e. that all bodies are built up of separate molecules, such molecules being either systems of vortex rings or groups of atoms of some other kind.

It is supposed that these molecules are in a state of constant motion, the mean velocity being dependent upon the temperature, and that the state of things may vary pretty continuously from the ultra-gaseous condition of Crookes, in which the mean free paths of the molecules are not only large compared with the dimensions of the molecules but may extend to a considerable number of millimetres, to the solid condition in which the molecules are not only always subject to the influence of neighbouring molecules but oscillate about positions of equilibrium.

We are in entire ignorance of the nature of the actions between the molecules of a solid. The fact that a solid may be converted into a liquid with a very slight change of volume, or even, as in the case of ice, with a diminution of volume, proves that the distinction between a liquid and a solid cannot depend solely upon the mean distance of the molecules. The production

* Fownes, *Organic Chemistry*, p. 417.

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or the disappearance of heat which accompanies a change of state would seem to indicate that the distinction does not depend solely on the mean velocity. The phenomena of diffusion shew that in a liquid which, apart from molecular motions, is at rest the molecules pass, though very slowly, from one part of the mass to another. In the case of a solid there is no evidence of what, when we compare its duration with the time taken by the molecule of a gas to describe its mean free path, we may call the secular motion of the molecules; we assume that they oscillate about positions of equilibrium, but of the forces which bind them together we know nothing.

The question then arises, what are the relations to one another of the molecules of a viscous body?

We may say generally that the viscous condition is one of transition between the liquid and solid states. A substance is sensibly viscous only through a certain range of temperature; if it is sufficiently cooled it becomes sensibly solid, if it is sufficiently heated it becomes a more or less limpid liquid. The range of temperature through which a body is sensibly viscous may be considerable, as in the case of pitch or shellac, or it may be very small, as seems to be the case with ice.

If we consider a portion of a viscous solid, however small, which is nevertheless large enough to be practically examined, it appears to be homogeneous; it does not follow however that if we were able to magnify it sufficiently to come within a "measureable distance" of being able to see the individual molecules it would still appear to be sensibly homogeneous. If it be allowable to apply the term "element" to a small parallelepiped whose dimensions are large compared with the dimensions of molecules but small compared with ordinary magnitudes, and which may contain (say) a few hundreds or thousands of molecules, we may say that it is quite conceivable that contiguous "elements" may be very different in their molecular architecture.

The hypothesis which I wish to suggest is that there is really such a distinction between different small portions of a viscous solid, and that we may describe the whole as consisting of a comparatively solid framework the interstices of which are filled by a more or less liquid mass. We may illustrate this conception by a sponge with its cavities filled with liquid, or better by a mass of gelatine which has imbibed water and passed into the ordinary condition of a jelly; the essential difference being that in these cases the solid framework and the liquid contents are of different chemical composition while in the supposed case of the viscous mass the "solid" and "liquid" parts consist of the same substance in different states of aggregation.

Of course any such picture of the molecular architecture of a

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viscous solid must be, so to speak, more or less "diagrammatic." In the present state of our ignorance as to the nature of the action between the molecules of a solid it would appear to be premature either to affirm or to deny that there is a distinct line of demarcation between what I have called the "solid" and what I have called the "liquid" parts of the mass. It might be more accurate to speak of the "more solid" and "less solid" portions; all that is involved in my supposition is that while some "elements" of the mass are in a state of aggregation which is essentially that of a solid others are in a state which more nearly resembles that of a liquid, and that we may pass from one extreme condition to the other in so small a space that the whole appears sensibly homogeneous when examined in any ordinary way. When I speak hereafter of the "solid" and "liquid" portions of the mass I must be understood to make use of the words subject to the foregoing qualifying remarks.

Now whenever we have a portion of any substance in contact with another portion of the same substance in a different state of aggregation we usually assume that there is at the common surface a condition of "mobile equilibrium;" that an interchange of molecules is going on between the portion in one state and the portion in the other state. The most familiar instance of this is when a volatile liquid is in contact with its own vapour in a confined space and at a constant temperature, but there is no reason to doubt that a similar state of things exist when a volatile solid is in contact with its own vapour or a fusible solid is in contact with its own liquid under analogous conditions.

In our supposed viscous body the relations between the "solid" portions and the contiguous "liquid" portions throughout the whole mass will be similar to those which obtain at the common surface of an ordinary fusible solid and its own liquid, and the most natural assumption seems to be that at each common surface throughout the mass there is a condition of "mobile equilibrium," so that an individual molecule which at one time formed part of a "liquid" portion of the mass may become attached to and form part of a "solid" portion and *vice versâ*. Subject to the same reservation as applies to the use of the words "solid" and "liquid" we may call those molecules which at any time form part of a "solid" element "attached" molecules and those which form part of a "liquid" element "unattached."

The peculiar characteristic of a viscous solid, viz. its gradual change of shape under the influence of external forces which are insufficient to produce a sensible change of shape in a very short time, follows as a natural consequence from the preceding supposition as to its molecular structure.

The first effect of an external force is to produce a slight and

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probably insensible deformation of the "solid" framework considered as an elastic solid. This will go on until the system of molecular stresses corresponding to the system of strains is in equilibrium with the external forces.

So far the case is parallel to what happens when external forces act upon a piece of jelly; but in the case of the jelly the position of equilibrium is permanent, the strained framework remains strained. In the case of the viscous body an interchange of molecules is constantly going on between the "solid" and the "liquid" portions of the mass; "attached" molecules are becoming "unattached" and "unattached" molecules are becoming "attached." This involves the replacement of strained "solid" elements by unstrained ones.

But with the complete or partial breaking up of a strained element a corresponding stress disappears simultaneously; there is no longer equilibrium between the external forces and the system of stresses; a further deformation of the mass will take place till equilibrium is restored; and so on continually. The successive deformations are cumulative while the stresses are not. The final change of form is the sum of the partial changes, but the sum of the molecular strains has a constant average value.

The rapidity with which the viscous body changes its form under the influence of given external forces will depend (1) upon the proportion of "liquid" to "solid" elements, which will affect the strength of the "framework" and therefore the amount of strain necessary to produce equilibrium, and (2) upon the rapidity with which the interchange of molecules between the "solid" and "liquid" portions of the mass takes place; and this again will depend partly upon the proportion of "unattached" to "attached" molecules and partly upon the mean average velocity of the molecules. On both these grounds the rapidity of the change of form will increase with the temperature.

(3) *Some applications of generalized space coordinates to differential analysis.* By Prof. J. LARMOR, M.A.

This paper is being printed in full in the *Transactions* of the Society.

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May 25, 1885.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

Prof. K. Pearson was elected a fellow of the Society.

The following communications were made to the Society:—

(1) *On the solution of the equations of vibrations of the ether and the stresses and strains in a light wave.* By R. F. GWYTHER, M.A. Communicated by Prof. J. J. THOMSON, M.A.

IN the first part of this paper I develop a solution of the equations relating to a light disturbance in a series of periodic terms with coefficients expressed in a manner comparable with the geometric series $\frac{\lambda}{r} \cdot \frac{\lambda^2}{r^2}$, etc., and am thus able to get a simple approximate form of solution suitable for distances from the source of light large compared with a wave length.

From this it follows that

$$d\xi/dx = x/ra \,.\, d\xi/dt, \text{ etc.},$$

so that the elements of the strain are proportional to the velocity of the displacement, and that at any particular place and time the kinetic and potential energies are equal, and their sum is variable. In this particular the disturbance is not analogous to a pendulum vibration.

At the same time, I find to what degree of accuracy the equation of continuity demands that the vibration shall be in the wave front.

In the next part of the paper I calculate the stresses of a second order in the medium due to the disturbance (the stresses of the first order being of a circular harmonic type do not contribute at all to the mean stresses). From this consideration I prove that, except near the point source, the mean stresses are such that $P_{xy} = P_{yx}$, et cetera. I find the expressions for the actual stresses and they turn out to be those required by Maxwell's Theory of the Electro-magnetic nature of light, that is they are of the same form as the stresses in the Electro-magnetic field. I also obtain Maxwell's Equations of Electro-magnetic force. Finally it would follow from this investigation that in a wave of plane polarised light the displacement is in the direction of the magnetic force.

I also discuss in the last part certain analogies which are of interest but do not satisfy the conditions developed in this paper.

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1. The equations of which the solution is required are

$$\frac{d^2\xi}{dt^2} = a^2\nabla^2\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and similar equations for η and ζ, where ∇^2 stands for the Cartesian operator $\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)$.

After solution I shall add the condition that

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0.$$

I shall suppose that the time enters the solution only through trigonometrical terms, and (in the first instance) that the disturbance arises from a single source at the origin. In this case the solution takes the form

$$\xi = \Sigma\{A_1 \sin p(r-at) + B_1 \cos p(r-at)\},$$

where $p=\frac{2\pi}{\lambda}$, λ being the wave length, and generally our unit for comparison of magnitudes.

I shall shew how A and B are to be expanded in series of terms descending in magnitude comparably with those of the geometrical series $1.\frac{\lambda}{r}.\frac{\lambda^2}{r^2}\ldots$ For on substituting this value of ξ in the differential equation, we get these equations of condition:

$$\left.\begin{aligned}\nabla^2 A_1 - \frac{2p}{r}\left\{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+1\right\}B_1 = 0\\ \nabla^2 B_1 + \frac{2p}{r}\left\{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+1\right\}A_1 = 0\end{aligned}\right\}\ldots\ldots(2).$$

Write now A and B in a series, in descending order, of homogeneous terms, beginning with one of degree -1, to ensure convergency, and use Euler's Theorem. Thus

$$A_1 = u_{-1}+u_{-2}+u_{-3}+\&c.$$

$$B_1 = v_{-1}+v_{-2}+v_{-3}+\&c.$$

$$\therefore \nabla^2\{u_{-1}+u_{-2}+\&c.\}+\frac{2p}{r}\{v_{-2}+2v_{-3}+\&c.\}=0,$$

$$\nabla^2\{v_{-1}+v_{-2}+\&c.\}-\frac{2p}{r}\{u_{-2}+2u_{-3}+\&c.\}=0.$$

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Hence u_{-1} and v_{-1} may be any homogeneous functions of x, y and z of degree -1, and the terms of other degrees are to be derived by the law

$$\left.\begin{aligned} &\frac{2p}{r}v_{-2}+\nabla^2 u_{-1}=0, \quad \frac{4p}{r}v_{-3}+\nabla^2 u_{-2}=0, \\ &\frac{2p}{r}u_{-2}-\nabla^2 v_{-1}=0, \quad \frac{4p}{r}u_{-3}-\nabla^2 v_{-2}=0, \end{aligned}\right\} \text{ &c. } \Big\} \ldots\ldots (3).$$

By this means we may obtain a complete formal solution suitable for values of r greater than λ, for on examination the orders of magnitude of the consecutive terms are to be compared with those of the geometric series $1 . \frac{\lambda}{r} . \frac{\lambda^2}{r^2}$, etc. Hence the series in the case of light may be considered to be limited to the first term, and would actually terminate provided $\nabla^2 u_{-n}=0$.

In the general case we notice that if u_{-1} and v_{-1} are divided into any number of parts to suit any linear algebraic or linear differential condition; and if each of these parts is expanded by the law above, the sum of the separate expansions will be the original expansion.

Thus if we write $\xi=\xi_0+k\left(\frac{d\zeta_0}{dy}-\frac{dy_0}{dz}\right)$, etc., we should find that for any value of k this division can be always made, and in one way only.

If, moreover, we want two similar expansions for η and ζ, where ξ, η, ζ are to satisfy the relation

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0,$$

we may obtain them without loss of generality, thus

$$\xi=\frac{d\zeta_0}{dy}-\frac{d\eta_0}{dz}, \qquad \eta=\frac{d\xi_0}{dz}-\frac{d\zeta_0}{dx}, \qquad \zeta=\frac{d\eta_0}{dx}-\frac{d\xi_0}{dy},$$

where ξ_0, η_0, ζ_0 are unconditional solutions

$$\xi_0=\Sigma\{A_1 \sin p(r-at)+B_1 \cos p(r-at)\},$$
$$\eta_0=\Sigma\{A_2 \sin p(r-at)+B_2 \cos p(r-at)\},$$
$$\zeta_0=\Sigma\{A_3 \sin p(r-at)+B_3 \cos p(r-at)\},$$

and
$$\xi=\Sigma\{C_1 \sin p(r-at)+D_1 \cos p(r-at)\}.$$

I have shewn how to deduce A and B from the first terms (say a and b). I will shew how to obtain the corresponding first terms in C and D (say c and d). The following terms will be found by the same rule as before.

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The first terms will evidently be obtained by differentiating only the trigonometrical parts. Thus

$$c_1 = -\frac{p}{r}(b_3 y - b_2 z), \qquad d_1 = \frac{p}{r}(a_3 y - a_2 z),$$

and similarly we may deduce the other terms.

For example,

$$d_1' = \frac{p}{r}(a_3' y - a_2' z) + \frac{db_3}{dy} - \frac{db_2}{dz},$$

where a', b', c', d' stand for the second terms in A, B, C, D.

If we proceed to operate upon ξ, η, ζ we may obtain a second derived vector ξ_2, η_2, ζ_2 which will have the form

$$\xi_2 = -\Sigma\Sigma\, p^2 \left\{\left(a_1 - x\frac{a_1 x + a_2 y + a_3 z}{r^2}\right)\sin p(r - at) + \left(b_1 - x\frac{b_1 x + b_2 y + b_3 z}{r^2}\right)\cos p(r - at)\right\},$$

where $\Sigma\Sigma$ relates to the summation for different values of p and also for the completion of the coefficients of the several terms.

This will become $\xi_2 = -\Sigma p^2 \xi_0$, provided

$$\left.\begin{aligned} a_1 x + a_2 y + a_3 z &= 0 \\ b_1 x + b_2 y + b_3 z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which form the conditions that $\frac{d\xi_0}{dx} + \frac{d\eta_0}{dy} + \frac{d\zeta_0}{dz} = 0$. Operate upon the first of these with ∇^2; write a' and b' for the second terms in A and B, and observe the equations (3) by which these are to be found.

$$\therefore\ x\nabla^2 a_1 + y\nabla^2 a_2 + z\nabla^2 a_3 + z\left(\frac{da_1}{dx} + \frac{da_2}{dy} + \frac{da_3}{dz}\right) = 0,$$

$$\therefore\ xb_1' + yb_2' + zb_3' = \frac{r}{p}\left(\frac{da_1}{dx} + \frac{da_2}{dy} + \frac{da_3}{dz}\right).$$

Now this is of the order λ/r compared with a and b, and therefore to this degree of approximation we may say that

$$x\xi + y\eta + z\zeta = 0,$$

or that the displacement lies in the wave front (Stokes' papers, Vol. II., On the Dynamical Theory of Diffraction, Art. 27, p. 275).

A proof that $x\xi + y\eta + z\zeta$ is approximately zero, on the condition that $\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0$ may be given in this way,

$$\nabla^2(x\xi + y\eta + z\zeta) = x\nabla^2\xi + y\nabla^2\eta + z\nabla^2\zeta,$$

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(on the condition that $\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$)

$$=\frac{1}{a^2}\frac{d^2}{dt^2}(x\xi+y\eta+z\zeta).$$

Hence $x\xi+y\eta+z\zeta$ satisfies the fundamental equation of this paper and can be expanded in the same form as ξ, η, ζ are themselves expanded, but the terms of order -1 in ξ, η, ζ will appear as of order 0 in $x\xi+y\eta+z\zeta$, and therefore the sum of such terms will vanish. We may use a convenient notation by writing this

$$\left.\begin{aligned} x\,_\xi u_{-1}+y\,_\eta u_{-1}+z\,_\zeta u_{-1}&=0\\ x\,_\xi v_{-1}+y\,_\eta v_{-1}+z\,_\zeta v_{-1}&=0\end{aligned}\right\}.$$

That $x\xi+y\eta+z\zeta$ may vanish absolutely will require only that the terms of order -2 should vanish, or that

$$\left.\begin{aligned} x\,_\xi u_{-2}+y\,_\eta u_{-2}+z\,_\zeta u_{-2}&=0\\ x\,_\xi v_{-2}+y\,_\eta v_{-2}+z\,_\zeta v_{-2}&=0\end{aligned}\right\}.$$

These conditions reduce to

$$\left.\begin{aligned} \frac{d}{dx}\xi u_{-1}+\frac{d}{dy}\eta u_{-1}+\frac{d}{dz}\zeta u_{-1}&=0\\ \frac{d}{dx}\xi v_{-1}+\frac{d}{dy}\eta v_{-1}+\frac{d}{dz}\zeta v_{-1}&=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5).$$

In ordinary cases of the propagation of light waves, we may neglect all parts of ξ, η, ζ except the greatest, and in finding the differential coefficients we need only to differentiate the trigonometrical function.

In this case we have

$$\frac{d\xi}{dx}=\frac{x}{ra}\frac{d\xi}{dt}, \qquad \frac{d\xi}{dy}=\frac{y}{ra}\frac{d\xi}{dt}, \qquad \frac{d\xi}{dz}=\frac{z}{ra}\frac{d\xi}{dt}, \qquad \frac{d\xi}{dr}=\frac{1}{a}\frac{d\xi}{dt} \quad \ldots(6),$$

and the variation of the displacement in directions perpendicular to the wave normal, or in the wave front, i.e. the elongation in lines in the wave front, is zero. This indicates that the components of the strain are greatest when the velocity is greatest, and that at any point the kinetic and potential energies have their zero and maximum values simultaneously.

By the relations already established, simple proofs can be given that the kinetic energy and the intensity (measured by $\xi^2+\eta^2+\zeta^2$) of the disturbance are propagated by the same law as the disturbance itself except near the origin of light. That is that

$$\left\{\frac{d^2}{dt^2}-a^2\nabla^2\right\}\{\xi^2+\eta^2+\zeta^2\}=0,$$

and

$$\left\{\frac{d^2}{dt^2}-a^2\nabla^2\right\}\{\dot\xi^2+\dot\eta^2+\dot\zeta^2\}=0.$$

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It will be sufficient to prove the first,

$$\frac{d^2}{dx^2}\xi^2 = 2\left(\frac{d\xi}{dx}\right)^2 + 2\xi\frac{d^2\xi}{dx^2},$$

using this and the analogous relations

$$a^2\nabla^2(\xi^2+\eta^2+\zeta^2) = 2a^2\left\{\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2 + \xi\nabla^2\xi + \text{etc.}\right\}$$

$$= 2\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 + \xi\frac{d^2\xi}{dt^2} + \eta\frac{d^2\eta}{dt^2} + \zeta\frac{d^2\zeta}{dt^2}\right\}$$

$$= \frac{d^2}{dt^2}(\xi^2+\eta^2+\zeta^2).$$

By the consideration of the dimensions it is plain that this equation is not accurate, since in that case the intensity would be expanded in a series similar to that for ξ, &c., in a manner corresponding to that used for $x\xi + y\eta + z\zeta$.

2. The order of magnitudes which we are considering compared with the amplitude of the disturbance, which we shall call simply d, is represented by

$$\xi : \frac{d\xi}{dt} : a\frac{d\xi}{dt} = \lambda : a : a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and therefore

$$\frac{d^2\xi}{dt^2} : \left(\frac{d\xi}{dt}\right)\frac{d^2\xi}{dxdt} : a^2\left(\frac{d\xi}{dx}\right)\frac{d^2\xi}{dx^2} = \lambda : \xi : \xi \ldots\ldots(8).$$

Hence, in neglecting terms in the equations of motion of an elastic solid which contain squares and products of the differential coefficients, we assume that ξ/λ may be neglected while our only declared rule has been to neglect λ/r; which is really treating them as comparable quantities. (With regard to this ratio ξ/λ, see Sir W. Thomson's paper quoted above, where it may be taken as not greater than 1 : 300.) In what follows I shall suppose that we may retain ξ/λ but neglect the square.

We have thus to extend our equations by including terms such as appear in the hydrodynamical equations and in the general theory of elasticity. If, in fact, a theory of molecular vortices, such as is touched upon by Maxwell (*Electricity and Magnetism*, Vol. II. Art. 823), is to be considered at all, the terms of the second degree upon which the theorems about vortices depend must be retained in the equations. I shall prove that the expressions already found for ξ, η, ζ identically satisfy the complete differential equation, a result which would be necessary if we are not to introduce terms affecting waves of half the length or say octaves of the original wave length.

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3. As we are to retain the squares and products of the differential coefficients of ξ, η, ζ, we must extend the expression for the work (w) required to produce the change of state of strain per unit volume. We therefore treat w as a cubic function of the roots of

$$\begin{vmatrix} K-A, & c, & b \\ c, & K-B, & a \\ b, & a, & K-C \end{vmatrix} = 0,$$

(Thomson and Tait, Vol. I. part 2. App. C. K.)

where $A = 1 + 2e$, etc., with the usual notation.

We will therefore write (in order to obtain all the terms of the second order)

$$w = \omega_2 + \omega_3 + \omega_3' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where

$$2\omega_2 = -n\{4fg + 4ge + 4ef \mp a^2 \mp b^2 - c^2\},$$

$$2\omega_3 = l\{4efg + abc - a^2e - b^2f - c^2g\},$$

$$2\omega_3' = m(e + f + g)\,\omega_2,$$

neglecting higher powers of the components of the strains.

From these we find the P's by the usual formulæ. I shall write down these formulæ for P_{xx}, P_{xy}, and P_{xz}, omitting the part of the first degree, which has been already employed in forming the equations and which being of the circular harmonic type contributes nothing to the mean pressure.

Then

$$P_{xx} = \frac{d\omega_2}{de}\cdot\frac{d\xi}{dx} + \frac{d\omega_2}{db}\cdot\frac{d\xi}{dz} + \frac{d\omega_2}{dc}\cdot\frac{d\xi}{dy} + \frac{d}{de}(\omega_3 + \omega_3'),$$

$$P_{xy} = \frac{d\omega_2}{df}\cdot\frac{d\xi}{dy} + \frac{d\omega_2}{da}\cdot\frac{d\xi}{dz} + \frac{d\omega_2}{dc}\cdot\frac{d\xi}{dx} + \frac{d}{dc}(\omega_3 + \omega_3'),$$

$$P_{xz} = \frac{d\omega_2}{dg}\cdot\frac{d\xi}{dz} + \frac{d\omega_2}{da}\cdot\frac{d\xi}{dy} + \frac{d\omega_2}{db}\cdot\frac{d\xi}{dx} + \frac{d}{db}(\omega_3 + \omega_3'),$$

and similar equations.

Before simplifying these, I shall limit the cases to those in which $P_{xy} = P_{yx}$, to which the special case I am considering belongs. The general condition for this is that the complementary minors of the determinant $\frac{d(\xi, \eta, \zeta)}{d(x, y, z)}$ may separately vanish, from which it follows that the leading minors vanish also.

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For $P_{xy} - P_{yx}$

$$= \left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right)\left\{\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right\} + \frac{d(\xi, \zeta)}{d(x, y)} - \frac{d(\eta, \zeta)}{d(z, x)} = 0.$$

Hence
$$\frac{d(\xi, \zeta)}{d(x, y)} = \frac{d(\eta, \zeta)}{d(z, x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

And these minors must be each 0, for otherwise the Jacobian above could be proved symmetrical, and therefore the disturbance would be without rotation, contrary to our hypothesis.

As a consequence of these conditions we have

$$\frac{d\xi}{dx} : \frac{d\xi}{dy} : \frac{d\xi}{dz} = \frac{d\eta}{dx} : \frac{d\eta}{dy} : \frac{d\eta}{dz} = \frac{d\zeta}{dx} : \frac{d\zeta}{dy} : \frac{d\zeta}{dz} = 1 : k_1 : k_2 \ldots(11).$$

We use these relations in simplifying P_{xx}, P_{xy}, P_{xz}, etc.

Thus

$$P = n\left\{2e^2 + b\frac{d\xi}{dz} + c\frac{d\xi}{dy}\right\} + l(4fg - a^2) + m\omega_2$$

$$= n\left\{\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2\right\} - l\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)^2 + m\omega_2 \quad \ldots\ldots(12),$$

$$P_{xy} = n\left\{2f\frac{d\xi}{dy} + a\frac{d\xi}{dz} + b\frac{d\xi}{dx}\right\} + l(ab - 2cg)$$

$$= n\left\{\frac{d\xi}{dx}\cdot\frac{d\eta}{dx} + \frac{d\xi}{dy}\cdot\frac{d\eta}{dy} + \frac{d\xi}{dz}\cdot\frac{d\eta}{dz}\right\} - l\left\{\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right\}\left\{\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right\} \ldots(13),$$

with similar quantities which can now be written down.

Before proceeding we will evaluate ω_2, ω_3, and ω_3'.

First,

$$2\omega_2 = n\{a^2 + b^2 + c^2 + 2e^2 + 2f^2 + 2g^2\}$$

$$\left\{2\left(\frac{d\xi}{dx}\right)^2 + 2k_1^2\left(\frac{d\eta}{dx}\right)^2 + 2k_2^2\left(\frac{d\zeta}{dx}\right)^2 + \left(k_1\frac{d\zeta}{dx} + k_2\frac{d\eta}{dx}\right)^2 + \left(k_2\frac{d\xi}{dx} + \frac{d\zeta}{dx}\right)^2 + \left(\frac{d\eta}{dx} + k_1\frac{d\xi}{dx}\right)^2\right\}$$

$$= n\left\{\left(\frac{d\xi}{dx} + k_1\frac{d\eta}{dx} + k_2\frac{d\zeta}{dx}\right)^2 + \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2 + \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 + \left(\frac{d\eta}{dz}\right)^2 + \left(\frac{d\zeta}{dx}\right)^2 + \left(\frac{d\zeta}{dy}\right)^2 + \left(\frac{d\zeta}{dz}\right)^2\right\}$$

$$= n\left\{\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2 + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).\right.$$

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We may obtain another expression for this by comparing with (12), thus

$$2\omega_2 = n\{a^2 - 4fg + b^2 - 4ge + c^2 - 4ef\}$$

$$= n\left\{\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)^2 + \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right)^2 + \left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)^2\right\} \ldots\ldots (15).$$

The first of these forms simplifies in the case we are specially treating, for by (6)

$$\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2 = \frac{1}{a^2}\left(\frac{d\xi}{dt}\right)^2, \text{ etc.,}$$

and

$$n = a^2.$$

$$\therefore\ \omega_2 = \tfrac{1}{2}\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\}.$$

As we have been treating the unit of volume as being the unit mass we may say that ω_2 is equal to the kinetic energy per unit of volume, and that actually and not on an average. The energy is not oscillatory between the wholly kinetic and the wholly potential as in the case of a pendulum, but when the kinetic energy is zero then ω_2 is zero. If it were not so we might expect that there would be a luminous trace of the path of light owing to the dying away of the oscillations, whereas this expression shews that the light passes and leaves the medium at rest without trace of its passage.

We have also the formula

$$\frac{a^2}{2}\left\{\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)^2 + \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right)^2 + \left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)^2\right\}$$

$$= \tfrac{1}{2}\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\} = \omega_2,$$

which I shall refer to later.

Secondly, to evaluate ω_3, write it as a determinant and border it thus, using the relations (11),

$$2\omega_3 = l \begin{vmatrix} 1, & 0, & 0, & 0, \\ \frac{d\xi}{dx}, & 2\frac{d\xi}{dx}, & \frac{d\eta}{dx} + k_1\frac{d\xi}{dx}, & k_2\frac{d\xi}{dx} + \frac{d\zeta}{dx} \\ \frac{d\eta}{dx}, & \frac{d\eta}{dx} + k_1\frac{d\xi}{dx}, & 2k_1\frac{d\eta}{dx}, & k_1\frac{d\zeta}{dx} + k_2\frac{d\eta}{dx} \\ \frac{d\zeta}{dx}, & k_2\frac{d\xi}{dx} + \frac{d\zeta}{dx}, & k_1\frac{d\zeta}{dx} + k_2\frac{d\eta}{dx}, & 2k_2\frac{d\zeta}{dx} \end{vmatrix}$$

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$$= l \begin{vmatrix} 1, & -1, & -k_1, & -k_2 \\ \frac{d\xi}{dx}, & \frac{d\xi}{dx}, & \frac{d\eta}{dx}, & \frac{d\zeta}{dx} \\ \frac{d\eta}{dx}, & k_1\frac{d\xi}{dx}, & k_1\frac{d\eta}{dx}, & k_1\frac{d\zeta}{dx} \\ \frac{d\zeta}{dx}, & k_2\frac{d\xi}{dx}, & k_2\frac{d\eta}{dx}, & k_2\frac{d\zeta}{dx} \end{vmatrix}$$

which is easily seen to be zero, and ω_3' is also evidently zero.

Thus the parts of the stress due to ω_3 and ω_3' do not themselves vanish under the conditions

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$$

and $P_{xy}=P_{yx}$, etc., but no work is required to produce the corresponding strains.

We may however find a valid reason for omitting these terms of a third order, from the consideration that if we suppose the expressions for ξ, η, ζ such that our equations if written in full are applicable in the immediate neighbourhood of the source itself, the terms contained in ω_3 would be the ruling terms and would as they are of an odd order be liable to a change of sign, and hence would be incompatible with stability of the medium. I shall therefore consider that these terms may be omitted from our equations.

4. I shall now proceed to analyze these stresses, and as a preliminary I shall shew that the vectors,

$$\frac{d\xi}{dt}, \quad \frac{d\eta}{dt}, \quad \frac{d\zeta}{dt},$$

and

$$\frac{d\zeta}{dy}-\frac{d\eta}{dz}, \quad \frac{d\xi}{dz}-\frac{d\zeta}{dx}, \quad \frac{d\eta}{dx}-\frac{d\xi}{dy},$$

are at right angles, for

$$\frac{d\xi}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\frac{d\eta}{dt}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\frac{d\zeta}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right) \quad \ldots\ldots(16)$$

evidently vanishes by (6).

Take first the part of the P's due to ω_2, and for simplicity take a point in the axis of x distant r from the origin as the point considered, and the axis of y' parallel to the momentary resultant of $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$, that of z' parallel to the resultant of

$$\frac{d\zeta}{dy}-\frac{d\eta}{dz}, \quad \frac{d\xi}{dz}-\frac{d\zeta}{dx}, \quad \frac{d\eta}{dx}-\frac{d\xi}{dy}.$$

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Then $$\left.\begin{aligned} P_{rr} &= 0, & P_{r\gamma'} &= 0, & P_{rz'} &= 0 \\ P_{\gamma'\gamma'} &= \left\{\frac{d\eta'}{dt}\right\}^2, & P_{\gamma'z'} &= 0, & P_{z'z'} &= 0 \end{aligned}\right\} \omega_2 \quad \ldots\ldots\ldots\ldots (17).$$

This indicates a tension in the wave front in the direction of the resultant of

$$\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}.$$

Next, the part of the P's due to ω_3' is

$$P_{rr} = 0, \quad P_{r\gamma'} = 0, \quad P_{rz'} = 0,$$

$$P_{\gamma'\gamma'} = 0, \quad P_{\gamma'z'} = 0, \quad P_{z'z'} = -l\left\{\frac{d\eta'}{dx'} - \frac{d\xi'}{dy'}\right\}^2,$$

or a pressure (?) also in the wave front in the direction of the resultant of

$$\frac{d\zeta}{dy} - \frac{d\eta}{dz}, \frac{d\xi}{dz} - \frac{d\zeta}{dx}, \frac{d\eta}{dx} - \frac{d\xi}{dy}.$$

Whereas the part due to ω_3' is a tension (hydrostatic) equal in all directions and of magnitude

$$\frac{m}{2}\left\{\frac{d\eta'}{dt}\right\}^2.$$

If now we complete the equations of motion by adding the terms of the second order formed as in the case of Hydrodynamics and Elasticity, supposing the terms of the first order satisfied, we have

$$\frac{d\xi}{dt}\cdot\frac{d^2\xi}{dxdt} + \frac{d\eta}{dt}\cdot\frac{d^2\xi}{dydt} + \frac{d\zeta}{dt}\cdot\frac{d^2\xi}{dzdt}$$

$$= \frac{d}{dx}P_{xx} + \frac{d}{dy}P_{xy} + \frac{d}{dz}P_{xz} \quad \ldots\ldots\ldots\ldots (18),$$

with two similar equations.

It is plain that the expression on the left will vanish identically, and the terms on the right vanish identically so far as they depend upon ω_2 and ω_3, but the terms depending upon ω_3' will remain and cannot be made to vanish by any further condition respecting ξ, η, ζ already sufficiently restricted. The existence of such terms would require a corresponding force in the medium.

Having investigated the stresses corresponding to ω_3 and ω_3', I shall now neglect them, supposing that m and l are absolutely zero in accordance with reasons already given.

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If we take into account the expressions we have found (12) and (13) for the parts of the stresses depending on ω_2, we may put $d\xi/dx = x/ra \,.\, d\xi/dt$, etc., and $n = a^2$, thence we get

$$P_{xx} = \left(\frac{d\xi}{dt}\right)^2, \quad P_{xy} = \frac{d\xi}{dt} \cdot \frac{d\eta}{dt}, \quad P_{xy} = \frac{d\xi}{dt} \cdot \frac{d\zeta}{dt},$$

$$P_{yy} = \left(\frac{d\eta}{dt}\right)^2, \quad P_{yz} = \frac{d\eta}{dt} \cdot \frac{d\zeta}{dt}, \quad P_{zz} = \left(\frac{d\zeta}{dt}\right)^2.$$

It will be noticed that $P_{xy} = P_{yx}$, etc., only when the terms of order λ/r are neglected, so that in the neighbourhood of the source the stresses are not of this particular nature, and some natural phenomena (possibly magnetization) may be expected to shew itself as an accompaniment.

5. To compare this with Maxwell's Electro-magnetic Theory of Light, let us suppose

$$\nu \frac{d\xi}{dt} = \alpha, \quad \nu \frac{d\eta}{dt} = \beta, \quad \nu \frac{d\zeta}{dt} = \gamma \ldots\ldots\ldots\ldots (19),$$

therefore

$$\mu\nu \frac{d\xi}{dt} = \frac{dH}{dy} - \frac{dG}{dz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

etc.

In the first part of this paper I shewed that F, G, H can be found from linear equations. Knowing this we may find them more simply thus,

$$\mu\nu \frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) = -\nabla^2 F = -\frac{1}{a^2}\frac{d^2F}{dt^2},$$

therefore

$$\frac{dF}{dt} = -\mu\nu a^2 \left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)$$

$$= -\frac{4\pi}{k} f \text{ (by Maxwell's theory);}$$

also

$$k\mu a^2 = 1,$$

therefore

$$2\pi f = \nu\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) \text{ and } 2\pi u = \nu\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz}\right),$$

etc., etc.

On this theory then the magnetic force is proportional to the time variation of the actual displacement, and the electric displacement to the molecular rotations, and these two vectors have been shewn to be always at right angles. In an ordinary light wave the actual disturbance would be coincident neither with the magnetic force nor the electric displacement, but in a plane

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polarised wave the disturbance would be in the direction of the magnetic force. To make this agree with Maxwell's expressions for the energies we must give a special value to ν, thus

kinetic energy $= \frac{1}{2\nu^2}\{\alpha^2+\beta^2+\gamma^2\}$,

$$\omega = \frac{8\pi^2 a^2}{\nu^2}\{f^2+g^2+h^2\};$$

putting $\frac{1}{\nu^2} = \frac{\mu}{4\pi}$, we get

$$\left.\begin{aligned}\text{kinetic energy} &= \frac{\mu}{8\pi}(\alpha^2+\beta^2+\gamma^2)\\ \omega &= \frac{2\pi}{k}(f^2+g^2+h^2)\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

which agree with Maxwell's expressions for the Electro-kinetic and Electro-static energies.

And these have been shewn to be equal.

From the formulæ from which this was proved it is seen that the name potential energy is not quite applicable to the electrostatic energy, and that the whole energy must consist of the sum of the two energies.

Lastly, from the equations

$$\frac{d\xi}{dt}\cdot\frac{d^2\xi}{dxdt}+\frac{d\eta}{dt}\cdot\frac{d^2\xi}{dydt}+\frac{d\zeta}{dt}\cdot\frac{d^2\xi}{dzdt} = \frac{d}{dx}P_{xx}+\frac{d}{dy}P_{xy}+\frac{d}{dz}P_{xz},$$

etc.,

omitting, on the ground that ours is one of the simpler cases of electro-magnetic action, the fact that each side vanishes independently, we get

$$\frac{d\zeta}{dt}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)-\frac{d\eta}{dt}\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right) = \frac{d}{dx}P_{xx}-\tfrac{1}{2}\left\{\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right\}+\frac{d}{dy}P_{xy}+\frac{d}{dz}P_{xz},$$

etc.,

or $4\pi(v\gamma-\omega\beta)$

$$= \frac{d}{dx}\{\alpha^2-\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\}+\frac{d}{dy}(\alpha\beta)+\frac{d}{dz}(\alpha\gamma),$$

etc.....(22),

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which equations agree with those of Maxwell (Vol. II., Art. 643), and lead me to suggest that the true stress due to strain is the tension along the lines of magnetic force and that the hydrostatic pressure is due to the velocities. The stress being made up of two parts just as the energy is made up of two parts.

6. I have taken above the variations of the actual displacement as proportional to the magnetic force, so that the electromagnetic energy is proportional to the kinetic. But as this is an open question, I will work out some other analogies.

(1) Put $\nu\xi = F$, $\nu\eta = G$, $\nu\zeta = H$, that is ξ, η, ζ proportional to the components of the vector potential

$$\left.\begin{aligned} \mu\alpha &= \nu\left\{\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right\} \text{ etc.} \\ f &= -\frac{k\nu}{4\pi}\frac{d\xi}{dt} \text{ etc.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Hence kinetic energy $= \dfrac{8\pi^2}{k^2\nu^2}\{f^2 + g^2 + h^2\}$,

potential energy $= \dfrac{a^2\mu^2}{2\nu^2}\{\alpha^2 + \beta^2 + \gamma^2\}$.

Put $\dfrac{1}{\nu^2} = \dfrac{k}{4\pi}$, and

kinetic energy $= \dfrac{2\pi}{k}(f^2 + g^2 + h^2)$,

potential energy $= \dfrac{a^2\mu^2 k}{8\pi}(\alpha^2 + \beta^2 + \gamma^2)$

$= \dfrac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2)$.

The result of this analogy is to interchange the kinetic and potential energies, but the stresses in the medium would be along the lines of the electric displacement instead of magnetic force. The form of the vector here representing α might, at first sight, appear the more suitable, but the vector whose components are $\dfrac{d\zeta}{dy} - \dfrac{d\eta}{dz}$, etc., has in our case no more rotational property than has $(\dot{\xi}, \eta, \zeta)$ or $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$.

In neither case do (α, β, γ) or (f, g, h) appear to have the proper dimensions, but the relative dimensions are of course correct on either supposition. In order to make the dimensions correct,

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put m for the density of the medium, but still the dimensions will differ from those of the electro-magnetic system. Thus if

$$\alpha = \nu \frac{d\xi}{dt},$$

and α is of dimensions $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}]$,

then ν is of dimensions $[M^{\frac{1}{2}} L^{-\frac{3}{2}}]$.

Then as before the kinetic energy

$$= \frac{m}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) = \frac{m}{2\nu^2}(\alpha^2 + \beta^2 + \gamma^2) = \frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma),$$

and potential energy

$$= \frac{8\pi^2 m a^2}{\nu^2}(f^2 + g^2 + h^2) = \frac{2\pi}{k}(f^2 + g^2 + h^2),$$

with

$$\frac{m}{\nu^2} = \frac{\mu}{4\pi};$$

but μ is of [0] dimensions, therefore ν is of $[M^{\frac{1}{2}}]$, which is contrary to our requirements.

(2) Mr Glazebrook has worked out another analogy in which $\nu d\xi/dt = F$, etc., founded on equations into which the electromotive force at a point is introduced as an impressed force but not apparently related to the strain of the medium. So long as linear equations only are used it is probable that other analogies could be similarly worked out. Put

$$\left.\begin{aligned} &\nu\dot{\xi} = F, \text{ etc.,} \\ \text{therefore} \quad &\nu\ddot{\xi} = \frac{dF}{dt} = -\frac{4\pi f}{k}, \\ &\nu \frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) = \frac{dH}{dy} - \frac{dG}{dz} = \mu a, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(24).$$

We might by starting with the vector $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ have proved the equality of the quantities

$$(\ddot{\xi}^2 + \ddot{\eta}^2 + \ddot{\zeta}^2) \text{ and } a^2\left\{\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz}\right)^2 + \text{etc.}\right\};$$

that is,

$$\frac{16\pi^2}{k^2}(f^2 + g^2 + h^2) = \mu^2 a^2 \{\alpha^2 + \beta^2 + \gamma^2\},$$

$$\frac{2\pi}{k}(f^2 + g^2 + h^2) = \frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2);$$

but these expressions would no longer be the kinetic and potential energies of any actual motion.

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The stress would in this case be along the lines of the electromagnetic momentum. All that can be said in favour of this analogy is, that it shews such agreement as is proved in Mr Glazebrook's paper (*Phil. Mag.*, June, 1881).

(3) I shall work out one other case, because, though not at first sight a desirable one, it appears to find some support from Maxwell.

Put $\left.\begin{aligned} \nu\xi &= f \\ \nu\dot{\xi} &= u \end{aligned}\right\}$ .....(25),

therefore

$$\nu\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)=\frac{dh}{dy}-\frac{dg}{dz},$$

which plays no part in the electro-magnetic theory

$$\nu\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)=\frac{dw}{dy}-\frac{dv}{dz};$$

which is also without importance.

Let $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ and $(\dot{\xi}_0, \dot{\eta}_0, \dot{\zeta}_0)$ be the velocities of two separate disturbances in the medium, then the term which will appear in the kinetic energy in which these velocities are connected is $(\dot{\xi}\dot{\xi}_0+\dot{\eta}\dot{\eta}_0+\dot{\zeta}\dot{\zeta}_0)$.

Putting $\nu\dot{\xi}=u$, etc., taking the triple integral over all space and transforming, neglecting the surface integrals, we get

$$\frac{1}{\nu}\iiint(\dot{\xi}_0 u+\dot{\eta}_0 v+\dot{\zeta}_0 w)\,dxdydz$$

$$=\frac{4\pi}{\nu}\iiint\left\{\dot{\xi}_0\left(\frac{dv}{dy}-\frac{d\beta}{dz}\right)+\dot{\eta}_0\left(\frac{d\alpha}{dz}-\frac{dv}{dx}\right)+\dot{\zeta}_0\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}dxdydz$$

$$=\frac{4\pi}{\nu}\iiint\left\{\alpha\left(\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}\right)+\beta\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)+\gamma\left(\frac{d\dot{\eta}}{dx}-\frac{d\dot{\xi}}{dy}\right)\right\}dxdydz.$$

Comparing this with Maxwell's provisional theory of the magnetic action on light (*Electricity and Magnetism*, Vol. II. §§ 824, 825), we see that the latent analogy underlying his hypothesis is that which is given above.

On this analogy the electric displacement would be proportional to the actual displacement. The electro-magnetic momentum and magnetic induction would become mere mathematical expressions. The electro-magnetic and electro-static energies could still be proved equal, the electro-static energy being now proportional to the intensity of the light. The only merit of this supposition appears to be that it is the basis of Maxwell's speculations.

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(2) *Note on* Prof. ROWLAND'S *paper on spherical waves of light and the dynamical theory of diffraction* (*Phil. Mag.* Vol. XVIII., June, 1884). By R. T. GWYTHER, M.A. Communicated by Prof. J. J. THOMSON.

(3) *On plane waves of the third order in an isotropic elastic medium, with special reference to certain optical phenomena.* By Prof. K. PEARSON.

1. The object of the present paper is to consider a simple case of wave motion in an isotropic elastic medium, when the displacements are not considered so small that the cubes of the space variations may be neglected. In general the three body-equations for an isotropic solid each contain upwards of 80 terms when we do not neglect the squares of small quantities and over 200 when we retain the cubes, so that they appear to be extremely unmanageable. In the simple case we are about to consider we shall find that it is necessary to retain cubes in order to introduce any change into the equations for wave motion.

2. Let u, v, w be the displacements of the point xyz of the solid parallel to the axes.

Then the strain on a very small element of the solid at (xyz) is fully determined by the quantities $s_1, s_2, s_3, \alpha, \beta, \gamma$, where if PQ be any elementary line drawn from xyz, and having projections $\delta x, \delta y, \delta z$ on the axis which becomes PQ' after distortion,

$$PQ'^2 = PQ^2 + 2s_1\delta x^2 + 2s_2\delta y^2 + 2s_3\delta z^2 + 2\alpha\delta y dz + 2\beta\delta z dx + 2\gamma\delta x\delta y.$$

It is easily shewn that

$$\begin{aligned}
s_1 &= u_x + \tfrac{1}{2}(u_x^2 + v_x^2 + w_x^2),\\
s_2 &= v_y + \tfrac{1}{2}(u_y^2 + v_y^2 + w_y^2),\\
s_3 &= w_z + \tfrac{1}{2}(u_z^2 + v_z^2 + w_z^2),\\
\alpha &= w_y + v_z + u_y u_z + v_y v_z + w_y w_z,\\
\beta &= u_z + w_x + u_z u_x + v_z v_x + w_z w_x,\\
\gamma &= v_x + u_y + u_x u_y + v_x v_y + w_x w_y.
\end{aligned}$$

Further, W the work will be a function of the variables

$s_1, s_2, s_3, \alpha, \beta, \gamma \qquad = F(s_1, s_2, s_3, \alpha, \beta, \gamma)$, say.

3. Let us consider the case of a plane wave in which the vibrations are in the face, and suppose this face parallel to the plane of xy. Furthermore let us assume the same vibrations are taking place at every point of the face and that there is no

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compressional wave; then u, v will be functions of z only, and w will equal zero. Hence it follows that

$$s_1 = s_2 = \gamma = 0, \qquad s_3 = \tfrac{1}{2}(u_z^2 + v_z^2),$$
$$\alpha = v_z, \qquad \beta = u_z,$$
$$W = F(\alpha, \beta, s_3).$$

Lagrange's method gives us at once:

$$0 = \iiint \delta W \, dx dy dz + \rho \iiint \left(\frac{d^2 u}{dt^2} \delta u + \frac{d^2 v}{dt^2} \delta v \right) dx dy dz$$

to determine the vibrations. Or:—

$$0 = \iiint \left(\frac{dW}{ds_3} \delta s_3 + \frac{dW}{d\alpha} \delta\alpha + \frac{dW}{d\beta} \delta\beta \right) dx dy dz$$
$$+ \rho \iiint \left(\frac{d^2 u}{dt^2} \delta u + \frac{d^2 v}{dt^2} \delta v \right) dx dy dz.$$

The first integral may be written

$$\iiint \left(\frac{dW}{ds_3} \left\{ u_z \frac{d\delta u}{dz} + v_z \frac{d\delta v}{dz} \right\} + \frac{dW}{d\alpha} \frac{d\delta v}{dz} + \frac{dW}{d\beta} \frac{d\delta u}{dz} \right) dx dy dz.$$

Integrating by parts and neglecting the surface terms, we find

$$- \iiint \left[\delta u \frac{d}{dz} \left(\frac{dW}{ds_3} u_z + \frac{dW}{d\beta} \right) + \delta v \frac{d}{dz} \left(\frac{dW}{ds_3} v_z + \frac{dW}{d\alpha} \right) \right] dx dy dz,$$

or the body equations take the form:

$$\left. \begin{aligned} \frac{d}{dz} \left(\frac{dW}{ds_3} u_z + \frac{dW}{d\beta} \right) &= \rho \frac{d^2 u}{dt^2} \\ \frac{d}{dz} \left(\frac{dW}{ds_3} v_z + \frac{dW}{d\alpha} \right) &= \rho \frac{d^2 v}{dt^2} \end{aligned} \right\} \quad \text{..... (i).}$$

4. We must now determine the form of W. So far as terms of the second order are concerned W must be of the form

$$2W = \lambda s_3^2 + \mu (\alpha^2 + \beta^2 + 2 s_3^2).$$

It is needful however to consider possible cubic and certain quartic terms. The only terms of the 3rd and 4th order in the expression for the work which would not give rise to terms higher than the third in the differential equations are of the form

$$c s_3 (\alpha^2 + \beta^2) + d s_3 \alpha\beta + e (\alpha^3 + \beta^3) + f \alpha\beta (\alpha + \beta)$$
$$+ g (\alpha^4 + \beta^4) + h \alpha^2 \beta^2 + i (\alpha^2 + \beta^2) \alpha\beta.$$

If we turn the axes through any small angle $\delta\theta$, s_3 remains unchanged, α becomes $\alpha - \beta\delta\theta$, and β, $\beta + \alpha\delta\theta$. Hence, since this cannot change the form of W it easily follows that d, e, f are all zero and $h = 2g$.

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Thus W is of the form:

$$2W = \lambda s_s^2 + \mu(\alpha^2 + \beta^2 + 2s_s^2) + 2cs_s(\alpha^2 + \beta^2) + g(\alpha^2 + \beta^2)^2.$$

We acccordingly find

$$\frac{dW}{ds_s} = (\lambda + 2\mu)s_s + c(\alpha^2 + \beta^2),$$

$$\frac{dW}{d\alpha} = \mu\alpha + 2cs_s\alpha + 2g\alpha(\alpha^2 + \beta^2),$$

$$\frac{dW}{d\beta} = \mu\beta + 2cs_s\beta + 2g\beta(\alpha^2 + \beta^2).$$

Hence, substituting in (i) we have

$$\left.\begin{aligned} \frac{\nu}{3}\frac{d}{dz}\left[(u_x^2 + v_x^2)u_x\right] + \kappa^2\frac{d^2u}{dz^2} &= \frac{d^2u}{dt^2} \\ \frac{\nu}{3}\frac{d}{dz}\left[(u_x^2 + v_x^2)v_x\right] + \kappa^2\frac{d^2v}{dz^2} &= \frac{d^2v}{dt^2} \end{aligned}\right\} \quad \text{.....(ii),}$$

where $$\kappa^2 = \frac{\mu}{\rho} \text{ and } \frac{\nu}{3} = \frac{\lambda + 2\mu + 4(c+g)}{2\rho}.$$

5. We proceed to draw some inferences from these equations.

(*a*) A 'plane polarised' wave or one of the type

$$u = A\cos\frac{2\pi}{\lambda}(z - \kappa t), \qquad v = B\cos\frac{2\pi}{\lambda}(z - \kappa t),$$

is not an accurate representation of a possible wave motion in an elastic medium unless $\nu = 0$.

This involves $\lambda + 2\mu + 4(c+g) = 0.$

It seems to me possible that c may equal zero, and in this case we have

$$g = -\frac{\lambda + 2\mu}{4},$$

or the coefficient of the terms of the fourth order is *negative*. If we do not look upon c as zero, we have in our particular case, since $s_s = \frac{1}{2}(\alpha^2 + \beta^2)$, for the terms of the fourth order

$$2W_4 = (c+g)(\alpha^2 + \beta^2)^2,$$

and we again find the coefficient of the terms of the fourth order negative. We are thus led to a certain relation between the elastic coefficients of the square and quartic terms of the work, which must be satisfied, if a plane polarised wave is to be propagated through the medium. It is not impossible that such relation holds or very nearly holds for the ether, it would denote that the expression usually taken after Green for the work is *too great*. There

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appears to be no ground for supposing that the above relation however would hold for every medium through which transverse waves might be propagated, and the retention of the above cubic terms leads to interesting results, which we shall now consider.

(*b*) It is possible for a plane wave, given by equations (ii), to remain in a certain sense polarised in one plane. For it is only needful to put $v=0$, which is a possible solution of the second equation, and the first then involving only u becomes

$$\frac{\nu}{3}\frac{du_z^3}{dz}+\kappa^2\frac{d^2u}{dz^2}=\frac{d^2u}{dt^2},$$

or,
$$\nu\left(\frac{du}{dz}\right)^2\frac{d^2u}{dz^2}+\kappa^2\frac{d^2u}{dz^2}=\frac{d^2u}{dt^2}.$$

As a first approximation let us take a single term of the type

$$A\cos\frac{2\pi}{\lambda}(z-\kappa t).$$

Then

$$\left(\frac{d^2}{dt^2}-\kappa^2\frac{d^2}{dz^2}\right)u=-A^3\frac{16\pi^4}{\lambda^4}\nu\sin^2\frac{2\pi}{\lambda}(z-\kappa t)\cos\frac{2\pi}{\lambda}(z-\kappa t)$$

$$=-A^3\frac{16\pi^4}{\lambda^4}\nu\cos\frac{2\pi}{\lambda}(z-\kappa t)$$

$$+A^3\frac{16\pi^4}{\lambda^4}\nu\cos^3\frac{2\pi}{\lambda}(z-\kappa t)$$

$$=-A^3\frac{4\pi^4}{\lambda^4}\nu\cos\frac{2\pi}{\lambda}(z-\kappa t)+A^3\frac{4\pi^4}{\lambda^4}\nu\cos\frac{6\pi}{\lambda}(z-\kappa t).$$

Hence
$$u=A\cos\frac{2\pi}{\lambda}(z-\kappa t)+A^3\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t\sin\frac{2\pi}{\lambda}(z-\kappa t)$$

$$-A^3\frac{\pi^3}{\lambda^3}\frac{\nu}{3\kappa}t\sin\frac{6\pi}{\lambda}(z-\kappa t).$$

These results are of a rather remarkable character, the additional terms introduced contain t as a factor of the amplitude, and although these *may* only be the first terms of a series of powers of t which may not necessarily become infinite with the time, yet it would seem that a wave of velocity κ could not be propagated in the medium as a stable motion. There would arise superposed waves of increasing amplitudes of (i) the same wave length but retarded by a quarter of a wave length, (ii) of one-third the wave length.

Has anything of this kind been observed? It would suggest that if a chemical ray were selected and passed through a proper

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medium, it might after transition be found to be accompanied by a strong heat ray. Further, we might be led to question whether the heat of a very hot body be not due to its giving off at the same time a mass of chemical or even violet rays; whether with Mr Langley the 'real colour of the sun be not *blue*'?

On the other hand the occurrence of terms with an amplitude containing t might suggest the breaking up of the wave, or the impossibility of transmitting a wave of a velocity κ which is independent of the intensity and of the wave length.

6. (*c*) Let us look at the matter from a somewhat different standpoint, and assume

$$u = A \cos \frac{2\pi}{\lambda}(z - kt),$$

where k is not equal to κ.

We find

$$\frac{d^2u}{dt^2} - \kappa^2 \frac{d^2u}{dz^2} = -\frac{A^2 4\pi^4}{\lambda^4} \nu \cos \frac{2\pi}{\lambda}(z - kt) + \frac{A^2 4\pi^4}{\lambda^4} \nu \cos \frac{6\pi}{\lambda}(z - kt),$$

or

$$\frac{d^2u}{dt^2} - \left(\kappa^2 + \frac{A^2\pi^2}{\lambda^2}\nu\right) \frac{d^2u}{dz^2} = \frac{A^2 4\pi^4}{\lambda^4} \nu \cos \frac{6\pi}{\lambda}(z - kt).$$

Hence approximately

$$u = A \cos \frac{2\pi}{\lambda}(z - kt) - A^2 \frac{\pi^2}{\lambda^2} \frac{\nu}{3\kappa} t \sin \frac{6\pi}{\lambda}(z - kt),$$

where

$$k^2 = \kappa^2 + \frac{A^2\pi^2}{\lambda^2}\nu.$$

From this we can again draw curious deductions. If a wave be transmitted into a medium for which ν is not zero, its velocity will depend upon its wave length and its intensity. Its velocity increases with its intensity as seems natural. Let k' be the velocity of a wave in a second medium, and suppose $\frac{k}{k'} = \mu$, the coefficient of refraction, then

$$\mu = r + \frac{s}{\lambda^2} + \frac{s'}{\lambda^4} + \text{etc.},$$

where r, s, s', etc. are certain constants, of which $s, s' \ldots$ depend partly on the *intensity* of the wave*. It will be observed that this result

* 'It has, however, long been the opinion of some philosophers that there are rays of different colours which have the same degree of refrangibility, and that there are rays of the same colour with different degrees of refrangibility.' Airy, *Undulatory Theory*, p. 157. Will the refrangibility being a function of the *intensity* elucidate this?

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is deduced from the ordinary theory of elasticity and without any assumption as to the ratio of wave length to molecular distance. We remark in the second place that the wave of length λ will be accompanied by a wave of one-third that length, which has the intensity a function of the time. Here again we have only the first term of a series which does not *obviously* become infinitely great as t increases. If u represented a wave of light (not a chemical ray from beyond the violet) for example, the wave of one-third its length would not fall within the sensible spectrum, and hence there might be some difficulty in ascertaining whether the above anomalous term had a real existence. It can hardly be doubted however that if it does exist it ought to manifest itself in some manner, for it would seem to correspond to the breaking up in some fashion of a ray of light of a single wave length transmitted through an isotropic medium. An isotropic medium would seem in a certain sense to possess 'double refraction' for a selected ray, it divides the ray into two parts, one of which has one-third of the previous wave length.

7. If we take u of the most general form

$$= \Sigma A_n \cos n\,(z - k_n t),$$

the anomalous terms do not necessarily appear unless one of the k_n's $= \kappa$. The treatment of this general form involves considerable difficulties, which I postpone for the present, as it requires careful examination, being not unsuggestive for the problems of absorption.

8. (a) Let us return to equations (ii) and suppose v is not zero, but that to begin with

$$u = A \cos \frac{2\pi}{\lambda}(z - \kappa t), \qquad v = B \cos \frac{2\pi}{\lambda}(z - \kappa t).$$

We find to the second approximation

$$u = A \cos \frac{2\pi}{\lambda}(z - \kappa t) + A\,(A^2 + B^2)\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa} t \sin \frac{2\pi}{\lambda}(z - \kappa t) - A\,(A^2 + B^2)\frac{\pi^3}{\lambda^3}\frac{\nu}{3\kappa} t \sin \frac{6\pi}{\lambda}(z - \kappa t),$$

$$v = B \cos \frac{2\pi}{\lambda}(z - \kappa t) + B\,(A^2 + B^2)\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa} t \sin \frac{2\pi}{\lambda}(z - \kappa t) - B\,(A^2 + B^2)\frac{\pi^3}{\lambda^3}\frac{\nu}{3\kappa} t \sin \frac{6\pi}{\lambda}(z - \kappa t),$$

which corresponds of course to what we have considered above, i.e. a plane wave of vibrations all parallel to a fixed direction, in this case $\tan^{-1}\frac{B}{A}$ and of intensity $A^2 + B^2$.

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(*b*) Next let us suppose

$$u = A\cos\frac{2\pi}{\lambda}(z-\kappa t), \qquad v = B\sin\frac{2\pi}{\lambda}(z-\kappa t),$$

which would correspond to elliptic vibrations.

We find

$$u = A\cos\frac{2\pi}{\lambda}(z-\kappa t) + \frac{A(3A^2+B^2)}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t\sin\frac{2\pi}{\lambda}(z-\kappa t) - \frac{A(A^2-B^2)}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t\sin\frac{6\pi}{\lambda}(z-\kappa t),$$

$$v = B\sin\frac{2\pi}{\lambda}(z-\kappa t) - \frac{B(A^2+3B^2)}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t\cos\frac{2\pi}{\lambda}(z-\kappa t) - \frac{B(B^2-A^2)}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t\cos\frac{6\pi}{\lambda}(z-\kappa t).$$

Thus it seems that in an elastic medium such as we are considering a wave of elliptic vibrations is not a stable form of wave motion.

If we neglect the third terms in u and v (which in the case of light would correspond to a ray outside the visible spectrum), we find $\frac{u^2}{A^2}(1+Q^2)+\frac{2uv}{AB}(Q-P)+\frac{v^2}{B^2}(1+P^2)=(1+PQ)^2$,

where

$$P = \frac{3A^2+B^2}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t,$$

$$Q = \frac{3B^2+A^2}{3}\frac{\pi^3}{\lambda^3}\frac{\nu}{\kappa}t.$$

Hence, neglecting for a first approximation the coefficients of t^2, we may say that if a wave of elliptic vibrations be started in an isotropic elastic medium the vibrational ellipse will retain the same shape but rotate with constant angular velocity $\frac{2AB}{3}\frac{\pi^3}{\lambda^2}\frac{\nu}{\kappa}$ about its centre in direction *opposite* to that of the vibrational displacement. Further, the rate of rotation is obviously a function of the wave length.

If we consider terms of the order (amplitude)^5, we find that the axes of the ellipse themselves vary with the time and the rate of rotation is no longer constant.

If the wave be one of circular vibration $A=B$, $P=Q$ and we find that to the fourth power of the amplitude this form of wave motion is stable. In this case the anomalous terms of one-third the wave length disappear. Even when we consider terms of the

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order (amplitude)^5 the vibration remains circular although the radius then alters with the time. It must be noted however that there would be in this case superposed anomalous waves, for we should have to find u and v to a third approximation.

These results seem noteworthy, for it would appear that a wave of circular vibration is more stable than a wave of linear vibration, and that whereas a wave of linear vibration is accompanied by anomalous waves which rapidly rise into importance, these disappear in the case of a wave of circular vibration to a much higher degree of approximation. Further it would seem that, disregarding the anomalous wave, elliptic vibration with a slow uniform rotation of the ellipse may for a very considerable time be a stable form of wave motion in an isotropic elastic medium. These remarks again have interest for the undulatory theory of light.

9. Let us assume more generally that

$$u = A\cos(mz - nt), \qquad v = B\cos(mz - nt + \alpha),$$

Then we find:

$$\frac{d^2u}{dt^2} - \kappa^2\frac{d^2u}{dz^2} = -\frac{m^2\nu}{12}A(3m^2A^2 + 4p^2B^2)\cos(mz - nt)$$

$$+ \frac{m^4\nu}{4}A^3\cos 3(mz - nt)$$

$$+ \frac{m(m+2p)p^2\nu}{12}B^2A\cos(\overline{m+2p}\,z - \overline{n+2q}\,t + 2\alpha)$$

$$+ \frac{m(m-2p)p^2\nu}{12}B^2A\cos(\overline{m-2p}\,z - \overline{n-2q}\,t - 2\alpha),$$

whence it follows that

$$u = A\cos(mz - nt) - \frac{m^4\nu A^3}{24n}t\sin 3(mz - nt)$$

$$- \frac{m(m+2p)p^2\nu B^2A}{24(n+2q)}t\sin(\overline{m+2p}\,z - \overline{n+2q}\,t + 2\alpha)$$

$$- \frac{m(m-2p)p^2\nu B^2A}{24(n-2q)}t\sin(\overline{m-2p}\,z - \overline{n-2q}\,t - 2\alpha),$$

where

$$n^2 - \kappa^2m^2 = \frac{m^4A^2}{4} + \frac{m^2p^2B^2}{3}.$$

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Similarly:

$$v = B\cos(pz - qt + \alpha) - \frac{p^4\nu B^3}{24q}\, t \sin 3(pz - qt + \alpha)$$
$$- \frac{p(p+2m)m^2\nu A^2 B}{24(q+2n)}\, t \sin(\overline{p+2m}\, z - \overline{q+2n}\, t + \alpha)$$
$$- \frac{p(p-2m)m^2\nu A^2 B}{24(q-2n)}\, t \sin(\overline{p-2m}\, z - \overline{q-2n}\, t + \alpha),$$

where $$q^2 - \kappa^2 p^2 = \frac{p^4 B^2}{4} + \frac{p^2 m^2 A^2}{3}.$$

Let us write $$m = \frac{2\pi}{\lambda}, \qquad n = \frac{2\pi}{\lambda} k,$$
$$p = \frac{2\pi}{\lambda'}, \qquad q = \frac{2\pi}{\lambda'} k';$$

therefore $$k^2 = \kappa^2 + 4\pi^2\left(\frac{A^2}{4\lambda^2} + \frac{B^2}{3\lambda'^2}\right),$$
$$k'^2 = \kappa^2 + 4\pi^2\left(\frac{A^2}{3\lambda^2} + \frac{B^2}{4\lambda'^2}\right).$$

In our solution above we have endeavoured to find waves which might be propagated through an isotropic elastic medium in such fashion that although anomalous waves might arise there should not be any such waves of the same arguments as the principal waves. Let us see at what results we have arrived.

If two waves consisting of vibrations in planes at right angles be propagated through an isotropic elastic medium, then

(i) These waves will interfere with each other, that is to say, either will produce anomalous waves in the plane of the other. This production of anomalous waves exists whatever may be the relation between the wave lengths, or between the velocities of propagation of the principal waves.

(ii) The velocities of propagation of the principal waves are altered by their coexistence, and in a manner which depends not only on the wave lengths but on the intensities.

(iii) If the two principal waves have the same velocities of propagation then it is necessary that

$$\frac{A}{\lambda} = \frac{B}{\lambda'},$$

or the amplitudes must be in the ratio of the wave lengths. If this condition be not fulfilled, there must be anomalous waves of

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the same wave lengths as the principal waves introduced into their respective vibratory planes.

(iv) If the two principal waves have equal wave lengths, they will as a rule have different velocities of propagation, depending not only on their own, but on each others' intensities. The velocity of a wave in one plane is accelerated by the existence of a wave in a plane at right angles.

(v) Equal wave lengths and equal velocities of propagation necessitate equal amplitudes, but in this case there will be anomalous waves of the same wave length and velocity arising from the terms with the phases

$$\left\{\begin{matrix}\overline{m-2p}\,z-\overline{n-2q}\,t-2\alpha \\ \overline{p-2m}\,z-\overline{q-2n}\,t+\alpha\end{matrix}\right\}.$$

The above results might all be translated into theorems concerning plane polarised light, but it seems idle to restate in the language of optics theorems which may after all have no bearing upon that subject.

10. We have in the course of our work (Art. 3) supposed that $w=0$, or that there is no wave of normal vibration. It may be as well to inquire a little more fully into the legitimacy of this supposition. If w be not zero we must take

$$s_3=\tfrac{1}{2}(\alpha^2+\beta^2)+\sigma+\tfrac{1}{2}\sigma^2,$$

where $\sigma=w_z$ and the other strains remain unchanged (Art. 2).

Now the expression we obtained for the work in Art. 4 may be written

$$2W=\mu(\alpha^2+\beta)^2+\frac{\nu\rho}{6}(\alpha^2+\beta^2)^2,$$

where
$$\nu=\frac{\lambda+2\mu+4(c+g)}{4\rho},$$

if we substitute for s_3 in that expression $\frac{1}{2}(\alpha^2+\beta^2)$.

If σ be not zero we must take for W,

$$2W=\mu(\alpha^2+\beta^2)+\frac{\nu\rho}{6}(\alpha^2+\beta^2)^2+(\lambda+2\mu)\sigma^2+e\sigma^3+f\sigma^4$$
$$+(\lambda+2\mu+2c)\sigma(\alpha^2+\beta^2)+h\sigma^2(\alpha^2+\beta^2),$$

where c is the same constant as before. We may write

$$\lambda+2\mu+2c=2\epsilon,$$

$$\therefore\ \frac{dW}{d\sigma}=(\lambda+2\mu)\sigma+\tfrac{3}{2}e\sigma^2+2f\sigma^3+\epsilon(\alpha^2+\beta^2)+h\sigma(\alpha^2+\beta^2),$$

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and the equation for normal vibrations may be written

$$(\lambda+2\mu)\frac{d^2w}{dz^2}+3\left(e+2f\frac{dw}{dz}\right)\frac{dw}{dz}\frac{d^2w}{dz^2}+\epsilon\frac{d}{dz}(u_s^2+v_s^2)$$
$$+h(v_s^2+w_s^2)\frac{d^2w}{dz^2}+h\frac{dw}{dz}\frac{d}{dz}(u_s^2+v_s^2)=\rho\frac{d^2w}{dt^2}.$$

The coefficient therefore of the term $\frac{d}{dz}(u_s^2+v_s^2)$ is $\epsilon+h\frac{dw}{dz}$, and *unless this vanishes there will be normal vibrations.* There are two important cases however in which it can be made to vanish. First, if $\sigma=-\frac{\epsilon}{h}$ a constant quantity. The expression for W will then be of the form $\mu'(\alpha^2+\beta^2)+\frac{\nu'}{6}(\alpha^2+\beta^2)^2$, where however the constants μ' and ν' will not be equal to the previous μ and ν. This result seems noteworthy. It would appear that: *if once a certain definite strain be given to the medium parallel to some straight line, then it is possible to send a wave of pure transverse vibrations in the direction of this strain.*

For in this case the equation for w is satisfied and we have similar equations to those of Art. 5 (*b*) for u and v.

This result suggests various inquiries: (*a*) as to whether a wave motion could be started which would produce or be accompanied by such a strain; (*b*) as to whether such states of strain permitting of transverse waves unaccompanied by normal waves may not exist for one or more directions in certain bodies.

The other case in which there would be no normal wave whatever is in a medium for which $\epsilon=0$. This involves

$$\lambda+2\mu+2c=0.$$

Hence we see that if the expression for the work contains no terms of the third order ($\epsilon=0$ and $e=0$)—a by no means improbable supposition—then there will be to a high degree of approximation no normal wave.

Should $\epsilon=0$, our expression for ν reduces to $\frac{g}{\rho}-\frac{\lambda+2\mu}{4\rho}$ and thus involves only the hitherto unconsidered elastic constant g. If g might be neglected as small in any case, we have for ν the following physical meaning—it is $\frac{1}{4}$ of the squared velocity of propagation of waves of normal displacement in the medium.

The above equation for w would doubtless give not uninteresting results for the normal wave which must accompany one of transverse vibration.

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11. Another more general form of integral may be obtained for the equation

$$\frac{d^2u}{dz^2}\left[\kappa^2+\nu\left(\frac{du}{dz}\right)^2\right]=\frac{d^2u}{dt^2}$$

as follows.

A first integral is

$$\pm\int\sqrt{\nu p^2+\kappa^2}\,dp=q+C, \text{ where } p=\frac{du}{dz},\ q=\frac{du}{dt}.$$

Hence

$$u=\beta z+\int\sqrt{\nu\beta^2+\kappa^2}\,d\beta\,.\,t+\phi_1(\beta)$$

where $$z+\sqrt{\nu\beta^2+\kappa^2}\,t+\phi_1'(\beta)=0\Big\},$$

and $\phi_1(\beta)$ is an arbitrary function of β. Another solution is obtained by changing the sign of the root and introducing a second arbitrary function, thus:

$$u=\gamma z-\int\sqrt{\nu\gamma^2+\kappa^2}\,d\gamma\,.\,t+\phi_2(\beta),$$

where $$z-\sqrt{\nu\gamma^2+\kappa^2}\,t+\phi_2'(\gamma)=0.$$

These solutions correspond respectively to waves propagated in opposite directions, and it will accordingly be sufficient to consider one—say the first—of them. We must however note that they cannot be superposed since the equation is not linear, or *waves cannot be propagated in opposite directions without affecting each other.* Integrating by parts—

$$\int\sqrt{\nu\beta^2+\kappa^2}\,d\beta$$

$$=\beta\sqrt{\nu\beta^2+\kappa^2}+\tfrac{1}{2}\left[\frac{\kappa^2}{\sqrt{\nu}}\log\left(\sqrt{\nu}\beta+\sqrt{\nu\beta^2+\kappa^2}\right)-\beta\sqrt{\nu\beta^2+\kappa^2}\right]+C$$

$$=\beta\sqrt{\nu\beta^2+\kappa^2}+\text{function of } \nu\beta^2+\text{higher powers},$$

$$=\beta\sqrt{\nu\beta^2+\kappa^2}+\chi(\beta).$$

Thus $$u=\beta z+\beta\sqrt{\nu\beta^2+\kappa^2}\,t+\chi(\beta)\,t+\phi(\beta)$$

$$=\phi(\beta)-\beta\phi'(\beta)+\chi(\beta)\,t,$$

where $\chi(\beta)$ shall by a proper choice of C be taken to vanish if we put $\nu=0$, which amounts to neglecting $\left(\frac{du}{dz}\right)^2$. In this latter case we find

$$\phi'(\beta)+z+\kappa t=0,$$

and

$$u=F(z+\kappa t).$$

Thus in the case of a single term representing a wave motion

$$u=A\cos n(z+\kappa t)=\phi(\beta)-\beta\phi'(\beta).$$

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12. Let us suppose this form of solution retained when ν is not considered zero and endeavour to discover what changes are introduced. We take

$$A \cos n\,(z + \sqrt{\nu\beta^2 + \kappa^2}\, t) = \phi\,(\beta) - \beta\phi'\,(\beta),$$

$$\therefore\ A \cos n\,\phi'\,(\beta) = \phi\,(\beta) - \beta\phi'\,(\beta)$$

is an equation to discover the form of ϕ.

Put $X = \phi'\,(\beta)$,

$$\therefore\ A \cos nX = \int X d\beta - \beta X,$$

$$-nA \sin nX = X\frac{d\beta}{dX} - \beta - X\frac{d\beta}{dX}.$$

$$\therefore\ \beta = nA \sin nX.$$

This gives β in terms of $\phi'\,(\beta)$*.

Hence we may write

$$u = A \cos n\,(z + \sqrt{\nu\beta^2 + \kappa^2}\, t) + \chi\,(\beta)\,.\,t,$$

where $$\beta = -nA \sin n\,(z + \sqrt{\nu\beta^2 + \kappa^2}\, t).$$

The complete value of the term $\chi\,(\beta)$ is

$$\frac{\kappa^2}{2\sqrt{\nu}}\left[\log\,(\sqrt{\nu}\beta + \sqrt{\nu\beta^2 + \kappa^2}) - \frac{\sqrt{\nu}\beta}{\kappa^2}\sqrt{\nu\beta^2 + \kappa^2}\right] + C$$

$$= -\frac{\nu\beta^3}{3\kappa} + \text{etc.},$$

by a proper choice of C.

Now let us see what happens to β when t is indefinitely increased. Obviously $\sin n\,(z + \sqrt{\nu\beta^2 + \kappa^2}\, t)$ will not increase indefinitely. Hence as t grows large β remains finite.

Thus it follows that $\chi\,(\beta)\,t = -\dfrac{\nu\beta^3 t}{3\kappa}$ does become infinite with t.

Or:

The terms which give rise to the anomalous wave grow larger and larger as the time increases indefinitely and the motion departs more and more from the simple wave form

$$u = A \cos n\,(z + \kappa t).$$

The velocity of wave propagation k is given by

$$k^2 = \kappa^2 + \nu\beta^2 = \kappa^2 + \nu n^2 A^2 \sin^2 n\,(z + \sqrt{\nu\beta^2 + \kappa^2}\, t).$$

* More generally we might have taken

$$\phi\,(\beta) - \beta\phi'\,(\beta) = F\{-\phi'\,(\beta)\}.$$

We should have found

$$\beta = -F'\,(X)$$

and $u = F(X) + \chi\,(\beta)\,t$, where $X = z + \sqrt{\nu\beta^2 + \kappa^2}\, t$ and $\chi\,(\beta)$ has the value in text.

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This result is not quite in accordance with Art. 6, and might suggest that the method of approximation there adopted is not entirely satisfactory.

It will be observed that the present article answers the doubt raised as to the nature of the t which appeared as a factor of the amplitude in Arts. 5 and 6; it really marks a breaking up of the wave motion.

13. The above remarks do not pretend to be a complete discussion of the higher terms in the equations for plane wave motion in an elastic medium; they are merely intended to point out that interesting results closely bearing on the undulatory theory of light may possibly be deducible from these terms. That these terms would lead to some formula resembling Cauchy's has been suggested, but it does not appear that any investigation of these higher terms has hitherto been published. If these higher terms really explain the dispersion of light in a *transparent* medium, then the refractive index ought to vary with the intensity. This result seems so contrary to all previous experience that it is perhaps sufficient to justify our rejecting such an explanation of dispersion.

(3) *Preliminary note on the theory of explosions.* By R. THRELFALL, B.A.

AN attempt was made to account for some of the anomalous effects observed in explosions by Sir Frederick Abel.

In the well-known experiments with detonators composed of various explosives, Abel was led to imagine that the apparent selective efficiency of chosen substances when applied to explode one another might be accounted for on a hypothesis of "synchronous vibrations." It was pointed out that this hypothesis can have no possible physical meaning, unless the vibrations be supposed to take place in the ether; and an explanation was sought in the behaviour of the products of explosion as regards their motion in air, and in water. For this purpose the explosives were treated in groups arranged with respect to their supposed time of decomposition, and it was shewn that much would depend on the method of "break up" of the volume of gas set free by the explosion. The various ways in which the energy of an explosion might be transmitted through fluids were enumerated, and some stress was laid on the effects to be expected if the conditions of explosion were such as to lead to the production of vortex rings.

The necessity of clearly defining the meaning of the phrase "violence of explosion" was pointed out, and it was shewn that

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the "violence" as defined by $\frac{\text{increase of volume}}{\text{time of explosion}}$ would not represent the relative destructive effects of explosions in free air.

Various experimental methods of treating the question of "break up" were described, and further communication was reserved pending the result of experiments still in progress.

(4) *Observations and statistics.* By F. Y. EDGEWORTH. Communicated by J. W. L. GLAISHER, M.A.

[*Abstract.*]

THE paper begins with a classification of the different cases which the (two) chief problems denoted by the title present: eight different principles of division are laid down. For example one division is between the cases where the weights of facility-curves are given beforehand, and where they are to be inferred from the observations (cp. Glaisher, *Mem. Astron. Soc.* XL. p. 103). Under this head it is remarked that many writers seem unduly to assume the modulus as known in cases like the statistics of male : female births; when they treat the $m+n$ events as so many *independent* black and white balls drawn from an urn. The fluctuation of the ratio, as inferred from the facts, the returns, is often very different from that assigned by such a simple hypothesis.

Another distinction is between facility-curves (other than probability-curves) which are, or are not, finite. The writer offers a simple proof of the law of error in the former case, disproof in the latter case.

Another distinction is between (α) observations so numerous as to present by simple induction or inspection the law of their genesis (the method of Quetelet, Mr Galton with his quartiles, deciles, &c., Mr Airy in his determination of modulus, &c.), and (β) the case where the data are viewed as samples, from which we are to ascend by way of *inverse probability* to the genesis of the observations (the method pursued by, e.g., Merriman in finding modulus and mean).

Under the first heading, Mr Galton's method (*Phil. Mag.*, 1878) of finding the number of elements in a "binomial" is criticised.

Under the heading of inverse probability, it is attempted to examine its foundations: whether, e.g., in determining inversely the modulus we are to assume as having *à priori* equal probability the different values of c (the modulus), or of c^2, or of h $\left(=\frac{1}{c}\right)$, or of h^2—all plausible and inconsistent. There is reached a conclusion agreeing with Laplace's first principle (Introduction, *Theorie Anal.*).

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In the case of facility-curves which may be regarded as probability-curves whose modulus is not given, *nor known to be identical* (cp. Glaisher, *loc. cit.*), the writer recommends a limiting (derived) function of the equation of $(n-1)$ degree,

$$\frac{1}{x-x_1}+\frac{1}{x-x_2}+\frac{1}{x-x_3}=0,$$

where x_1, x_2, &c. are the observations.

In the general case of facility-curves not probability-curves, it is argued that, though a perfect solution of the problem: What is the best Mean is unattainable (the writer retracting his hasty statement to the contrary in *Phil. Mag.*, Feb. 1884), yet an *approximate* solution in the form of a weighted arithmetical mean is very generally afforded, if the facility-curves can be regarded as of the form

$$A\rho^{-Mx^2+Px^4\ldots+Rx^r},$$

by taking the $(r-1)^{\text{th}}$ differential of

$$\begin{cases} -\{M_1(x-x_1)^2+M_2(x-x_2)^2+\&c.\} \\ +\ P_1(x-x_1)^4+P_2(x-x_2)^4+\&c. \\ +\ R_1(x-x_1)^r+R_2(x-x_2)^r+\&c. \end{cases}$$

a 'limiting' function of the equation whose solution is required.

This method, imperfect as it is, preferred to the method of least squares, which is criticised at length. It is shewn that its fundamental principle is identical with what Mr Todhunter calls "assumed inversion" (Todhunter, *Prob.*, p. 566): namely, that we can test a Mean, or *method of reduction*, say $\theta(x_1, x_2\ldots)$, by putting ourselves as it were at the source of error, taking every set of values such as

$$\left.\begin{matrix}(1)\quad x_1,\ x_2,\ x_3 \\ (2)\quad x_1',\ x_2',\ x_3'\end{matrix}\right\},$$

which as it were emanate from the source according to the (supposed known) facility-curves of the observations, forming for each set the mean θ, e.g.

$$\left.\begin{matrix}(1)\quad \theta(x_1,\ x_2,\ x_3\ldots) \\ (2)\quad \theta(x_1',\ x_2',\ x_3'\ldots) \\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\end{matrix}\right\},$$

and then observing the divergence of the facility-curve presented by the values of θ. This principle, it is argued, is theoretically incorrect and practically leads to wrong conclusions, e.g. if it were proposed to find the best value of n in the function

$$\left\{\frac{x_1^{\frac{1}{n}}+x_2^{\frac{1}{n}}+\&c.+x_s^{\frac{1}{n}}}{S}\right\}^n,$$

considered as a *method of reduction.*

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Another distinction is between real and fictitious or subjective means (the latter belonging to statistics in so far as distinguished from observations). There is a real mean in the ordinary case of a physical quantity elicited from observations. A subjective mean is of this nature. In the case of fluctuating phenomena, e.g. prices, we may select a certain value (not as that of a real thing, but) as the best representative of the whole set; which, if we must put one for many, minimizes the detriment incidental to that necessity. The subjective mean is found by a mathematical process analogous to Laplace's reasoning at p. 333, *Theor. Anal.* 3rd ed. (p. 365, Nat. ed.). In the case of a simple facility-curve this mean is the central point, other in other cases.

Laplace's theory in the passage cited is defended as the most philosophical view of the problem of observations. In fact, though we begin with the search of a real point—namely that from which observations have emanated—we have to take as a proximate end a certain subjective mean. When we have found by inverse probabilities the relative frequency with which different points originate the given observations x_1, x_2, x_3..., we seek the *subjective mean* of the set of values found.

The nature of a subjective mean explains Laplace's conception of the "most advantageous" as distinguished from the "most probable" value.

It is attempted to elucidate many other vexed passages in *Theorie Analytique*, e.g. the method of situation, the proof offered in the second supplement of the accuracy of the method of least squares, the assumption of mean error as test of advantage, &c.

The paper is being printed in full in the Transactions of the Society.

Note on MR IBBETSON'S *paper "On the free small normal vibrations of a thin homogeneous and isotropic elastic shell, bounded by two confocal spheroids.* Communicated Jan. 28, 1884. Correction by the author.

The expression assumed on p. 70, line 19, for the thickness τ of the shell at (η, ζ), supposes the tangent planes at the points on the outer and inner surfaces of the shell having these coordinates to be parallel. It is in fact the formula appropriate to a shell bounded by *similar* spheroids.

The correct expression for τ for confocals is

$$\tau = \frac{\epsilon}{[h_1]} = \frac{\epsilon}{2ac} \cdot \eta^{\frac{1}{2}}.$$

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For obvious reasons this error has no effect on the apparent verification of the former results, by reduction to the case of concentric spheres.

The correction greatly simplifies the expressions for the potential energy, and they may now be integrated with comparative ease.

With our former notation we have, for the oblate shell

$$\left.\begin{aligned} T_1 &= \frac{\pi}{60} \cdot \frac{\rho a \beta^2 (3 + 4\alpha^2 + 8\alpha^4)}{\alpha^2} \cdot \dot{\xi}^2 \\ W_1 &= \frac{\pi}{192} \cdot \frac{q \beta^6 A_1}{(1-\mu^2)\, a\alpha^6} \cdot \xi^2 \end{aligned}\right\},$$

and for the prolate shell

$$\left.\begin{aligned} T_2 &= \frac{\pi}{60} \cdot \frac{\rho a \beta^2 (8 + 4\alpha^2 + 3\alpha^4)}{\alpha^2} \cdot \dot{\xi}^2 \\ W_2 &= \frac{\pi}{192} \cdot \frac{q \beta^6 A_2}{(1-\mu^2)\, a\alpha^6} \cdot \xi^2 \end{aligned}\right\},$$

where
$$\begin{aligned} A_1 = \frac{1}{\sqrt{1-\alpha^2}} \int_0^{\cos^{-1}\alpha} \{ & 9\cos^4\theta - 6\,[1 + (1-\mu)\,\alpha^2]\cos^2\theta \\ & + 1 + 2\,(1-4\mu)\,\alpha^2 + 2\,(1+2\mu)\,\alpha^4 \\ & + 2\alpha^2(1-\alpha^2)\,[\mu - (1-\mu)\,\alpha^2]\sec^2\theta \\ & + \alpha^4(1-\alpha^2)^2\sec^4\theta \}\, d\theta ; \end{aligned}$$

$$\begin{aligned} A_2 = \frac{1}{\sqrt{1-\alpha^2}} \int_0^{\cos^{-1}\alpha} \{ & 9\alpha^6\sec^5\theta - 6\alpha^6(1-\mu+\alpha^2)\sec^3\theta \\ & + \alpha^4\,[2\,(1+2\mu) + 2\,(1-4\mu)\,\alpha^2 + \alpha^4]\sec\theta \\ & + 2\alpha^2(1-\alpha^2)\,(1-\mu-\mu\alpha^2)\cos\theta \\ & + (1-\alpha^2)^2\cos^3\theta \}\, d\theta. \end{aligned}$$

Thus, on integration,

$$\begin{aligned} A_1 = {} & (\tfrac{17}{24} + 2\mu)\,\alpha - (\tfrac{11}{4} - 3\mu)\,\alpha^3 + (1-2\mu)\,\alpha^5 + \tfrac{2}{3}\alpha^7 \\ & + \frac{\cos^{-1}\alpha}{\sqrt{1-\alpha^2}} \left[\frac{11}{8} - (1+5\mu)\,\alpha^2 + 2\,(1+2\mu)\,\alpha^4 \right]; \end{aligned}$$

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$$A_2 = \tfrac{2}{3} + (1-2\mu)\,\alpha^2 - (\tfrac{11}{4} - 3\mu)\,\alpha^4 + (\tfrac{17}{24} + 2\mu)\,\alpha^6 + \frac{\log \dfrac{1+\sqrt{1-\alpha^2}}{\alpha}}{\sqrt{1-\alpha^2}} \left[2\,(1+2\mu)\,\alpha^4 - (1+5\mu)\,\alpha^6 + \frac{11}{8}\,\alpha^8 \right].$$

The times of vibration are

$$\left.\begin{aligned} t_1 &= \frac{8\pi a}{\beta^2} \sqrt{\frac{\rho\,(1-\mu^2)}{5q} \cdot \frac{\alpha^3\,(3+4\alpha^2+8\alpha^4)}{A_1}} \\ t_2 &= \frac{8\pi a}{\beta^2} \sqrt{\frac{\rho\,(1-\mu^2)}{5q} \cdot \frac{\alpha^3\,(8+4\alpha^2+3\alpha^4)}{A_2}} \end{aligned}\right\}.$$

To the first power of the ellipticity $e\,(=1-\alpha)$ of the meridional section,

$$\left.\begin{aligned} t_1 &= \frac{8\pi a}{\beta^2} \sqrt{\frac{3}{2} \cdot \frac{\rho\,(1-\mu)}{q}} \cdot (1-e) \\ t_2 &= \frac{8\pi a}{\beta^2} \sqrt{\frac{3}{2} \cdot \frac{\rho\,(1-\mu)}{q}} \cdot (1-2e) \end{aligned}\right\},$$

whence it appears that, in a series of shells of the same material, whose inner and outer surfaces have the same major axes (a and $a\sqrt{1+\beta^2}$), the sphere is the form of *maximum* period for the class of vibrations considered.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 26, 1885.

ANNUAL GENERAL MEETING.

THE following were elected officers and new members of the Council for the ensuing year:—

President:

Prof. Foster.

Vice-Presidents:

Prof. Liveing, Prof. Babington, Prof. Adams.

Treasurer:

Mr J. W. Clark.

Secretaries:

Mr Trotter, Mr Glazebrook, Mr Vines.

New Members of Council:

Prof. Stokes, Prof. Lewis, Prof. J. J. Thomson, Mr Larmor.

The list of names of Benefactors of the Society was read by the Secretary.

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The following were elected Associates:

Mr Robt. Bowes.
Mr J. Carter.
Mr A. Deck.
Capt. P. Going.
Mr W. Heape.
Mr R. T. Lynch.
Mr W. Marshall.
Mr H. Middleton.
Mr W. E. Pain.
Mr W. W. Smith.
Mr R. H. Solly.

The following communications were made to the Society:

On the measurement of kinetic molecular energy on an absolute scale. By G. D. Liveing, M.A., F.R.S.

In treatises on thermo-dynamics no definition is usually given of the particular molecular motions which are to be regarded as heat. At least two very different kinds of motion are included under the name heat by all writers on the subject, namely the motion in a gas which produces pressure and the motion which produces radiation. Most writers now-a-day distinguish between heat and the motions of liquefaction, vaporization, and that of decomposition of molecules, and such expressions as "latent heat of evaporation" are used under protest, or changes of state are reckoned as internal work. When heat is employed to do external work it is almost exclusively the motion of translation of the molecules of gases which is directly employed, but the energy is supplied to the working substance by conduction in the form of heat, and all the other forms of energy that I have mentioned may be drawn upon, even the energy of dissociated chemical elements if there are such in the supply. For the sake of distinctness in what follows I shall use the word "heat" to mean only those forms of energy which consist in the vibratory motions producing radiation and those which pass in conduction. I do not pretend to say that these two kinds of motion are identical, but merely that for the present I class them together as one form of energy which all, so far as I know, call "heat." The motions of liquefaction, vaporization, and of the breaking up of molecules into simpler parts I shall distinguish from heat as separate forms of kinetic molecular energy. Nevertheless when the supply from which heat is drawn to work an engine consists of any ordinary materials in a state of thermal equilibrium at first, then when heat is withdrawn from them an adjustment takes place between all the forms of kinetic molecular energy in the supply in accordance with the law of dissipation whereby, in general, they all ultimately contribute their quotas to make good in part

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the loss of heat, and in this way they may all be said to take part in the work done by the engine supplied by them. They all have their effect on the temperature of the supply on which the availability of its energy essentially depends, and it may be observed that in the category of forms of energy which thus contribute to maintain the temperature of the supply must be included the energy of dissociation, whether the dissociation consist in the breaking up of the molecules into parts which are homogeneous, as happens with sulphur vapour and many other vapours at high temperatures, or whether it is dissociation of chemically diverse kinds of material.

Now considering that all these several forms of energy contribute to the temperature of a body it seems hardly possible to avoid the conclusion that they are all subject to the thermodynamic laws and that the conclusions drawn from those laws are all applicable, mutatis mutandis, to them as well as to heat. Many authors appear to me to have made tacitly some such assumption, and I have myself done so avowedly in considering the influence of dissipation of energy in regard to Chemical Equilibrium[1]. It is possible however to follow out the reasoning with regard to each form of kinetic molecular energy in lines parallel to those employed by writers on thermo-dynamics.

The second law of thermo-dynamics is a particular case of the more general law of dissipation of energy. That law may be stated as follows:

Every change which takes place spontaneously in the form or distribution of energy in any given portion of matter without accession of energy from without reduces the availability of the energy of that matter to do mechanical work.

No machine self-acting can then convert energy from a form in which it is less available into one in which it is more available.

It follows then on Carnot's principle that if we have a reversible engine working between a supply of energy at one degree of availability and a sink at some very slightly lower degree of availability, the fraction of each unit of energy derived from the supply which can be converted by the engine into mechanical work will depend only on the degree of availability of the supply and on the difference of degrees of availability between the supply and sink.

The expression degree of availability is an awkward one and not always appropriate so that I prefer to use "potential" instead of it. The fundamental notion connected with equality of potential

[1] *Chemical Equilibrium the result of the dissipation of Energy.* Cambridge, 1885. Deighton, Bell and Co.

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being that when two forms of energy are at the same potential in the same substance there is no tendency for either to be increased at the expense of the other; but that if they are at unequal potentials there is a tendency to an equalization; also that when two bodies have their energies at equal potentials there is no tendency for the energy of one to increase at the expense of the other, while if they are at unequal potentials there is a tendency to equalization by the passage of energy from one body to the other.

It remains to construct a scale of measurement which shall be applicable to all kinds of kinetic molecular energy.

Let S and S_0 be two systems each homogeneous in regard to both matter and energy, and let E, E_0 be their respective energies in such forms e_0, e_1, e_2 &c. as are from the nature of the systems capable of being freely transformed into each other, and freely transmitted from S to S_0 or vice versa when S and S_0 are placed in communication; e_0 for instance may be heat, e_1 the energy of motion of translation of the molecules of a gas, e_2 the energy of chemical separation and so on. E will be the sum of $e_0 + e_1 + e_2 +$ &c. for S; and E_0 the corresponding sum for S_0. S and S_0 may have besides E and E_0 other energies which are not freely transformable into any of the forms included in E and E_0. If S and S_0 are each in equilibrium, e_0, e_1, e_2, &c. in S will all be at the same potential V, and e_0, e_1, e_2, &c. in S_0 will also all be at one potential V_0. Also if when S and S_0 are put in communication there is no tendency for any part of E to transfer itself to S_0, or for any part of E_0 to transfer itself to S, then $V = V_0$; but if part of E_0 passes to S, V_0 is greater than V, and if part of E passes to S_0, V is greater than V_0.

Suppose V greater than V_0 and the difference to be δV, then it follows, on Carnot's principle, that no more work can be got out of E in consequence of the equalization of the potentials of E and E_0 when S and S_0 are put into communication by any means than can be obtained by means of a perfect reversible engine working between S and S_0; and hence the greatest amount of work which can be got out of E by such means will depend only on V and δV when δV is very small.

Let us suppose S_0 to be maintained by external agency in a constant state, so that the potential of E_0 is always V_0 during the operation of transferring part of E to S_0 and converting the rest into mechanical work by help of the reversible engine. Then if δW be the greatest amount of work which can be got out of E by this means, we shall have

$$\delta W : E = \delta V : F(V) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

when $F(V)$ is some function of V.

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Now suppose the engine to be worked backwards so as to restore S to its original state, taking as much energy as possible from S_0; then δW is the least amount of work which must be done upon S to restore to it its original energy E at potential V, and therefore $E-\delta W$ is the amount of energy which must be supplied to S in addition to the work δW done upon it. $E-\delta W$ must therefore be the energy of S when at the potential V_0, because no energy could pass spontaneously from S_0 to S at a higher potential, and δW is the minimum amount of energy which must be derived from without to raise the potential from V_0 to V.

Next let us measure V on such a scale that δV bears a constant ratio to δW when E is increased or diminished; and let S_0 be so changed in its state by external agency that the potential of E_0 is reduced to $V_0-\delta V$; then by means of a perfect engine working between S which has energy $E-\delta W$ at potential V_0, and S_0 maintained with its energy E_0 at potential $V_0-\delta V$, a second quantum of work, equal to the former quantity δW since δV is the same and is always proportional to δW, can be got out of $E-\delta W$, and we shall have

$$\delta W : E-\delta W = \delta V : F(V_0)$$
$$= \delta V : F(V-\delta V),$$

but by equation (1)

$$\delta W : E-\delta W = \delta V : F(V)-\delta V,$$

whence $$F(V-\delta V) = F(V)-\delta V,$$

and $$F(V) = V,$$

and equation (1) becomes

$$\delta W : E = \delta V : V \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since the work that can be got out of each unit of energy in E is the same fraction of δW, that one unit of energy is of E we may write in the above equation $dW : dE$, the rate of work obtainable per unit of E, instead of $\delta W : E$, and make δV always in a constant ratio to $dW/dE \,.\, \delta E$, where δE is an arbitrary increment of E which may be the unit of energy. δV, the increment of V, is then wholly independent of the nature of S and of the amount of energy E, and we have a method of measurement of V applicable to all forms of kinetic molecular energy, and we can make the magnitude of our degrees of potential the same for all. When the energy in question is heat, and V is temperature, the equation (2) becomes that which is usually given in text-books for the relation between the amount of work which can be got out of a supply of heat E at temperature V on the absolute dynamic scale when the refrigerator is at temperature $V-\delta V$.

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If S_1 be some independent system with energy E_1, or if the nature or circumstances of S be such that it may have energy in some form E_1, which cannot freely be transformed into any of the forms of E, the potential of E_1 will be independent of that of E but may be measured on the same scale if δV is always in the same ratio to dW/dE_1.

If the scales of measurement of any two forms of energy E and E_1 coincide at one point they must coincide at all points. Coincidence at one point may be determined by the consideration that when E and E_1 exist in a system or in two communicating systems in such circumstances that the one can be transformed into the other, there will be no tendency to any such transformation when both are at the same degree of potential, for no work could be got out of any such transformation.

The zero point may be taken arbitrarily, and as a zero point has already been assigned to the dynamic scale of temperature the same point may be retained in all cases. It is plain that E and V will vanish together whenever E can be directly transformed into heat, and this appears to be the case whenever the energy is strictly speaking the energy of the molecules.

Molar energy does not exhibit differences of potential.

When the energy is what is called energy of position, that is when the energy is that of the field, it is only when there is some difference in the energy of the field in different parts occupied by the system that any dissipation of the energy of the field can occur within the system. The potential of the energy of the field at that point within the system where it is least will therefore coincide with the zero of our scale when the energy of the field has to be brought into comparison with the other forms of energy in the system.

When the energy of the field is gravitation we can hardly include in the field of any experiment a sufficient difference of gravitational potential to bring the energy of the field into comparison in a measurable way with the molecular energies of the materials of our experiments. Nevertheless in the large field of nature differences of gravitational potential must, I should think, be capable of producing such effects as chemical decomposition. For example, the stable arrangement of a mass of mixed hydrogen and oxygen under gravity is not one in which the proportions of the two gases are constant at all levels, and if we have a large mass of aqueous vapour extending to a considerable height above the earth, there must be a tendency on account of gravitation towards a separation of hydrogen and oxygen and an accumulation of hydrogen in the upper part of the mass, and of oxygen in the lower part. Unless the potential of the energy of chemical separation passes per

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saltum from zero to some amount greater than the difference of potential which can be established by the combined effect of gravitation and the other energies of the system within the space occupied by the molecule, one would suppose that dissociation must occur and by the action of gravitation be rendered permanent at least in part.

(2) *On the transit of Venus,* Dec. 6, 1882. By J. B. PEARSON, D.D.

The paper contains an account of the reduction of observations made by Dr Pearson, with a calculation of the value of the Solar Parallax, found by comparing these with the French observations.

November 9, 1885.

PROFESSOR FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:—

(1) *On Weissmann's New Theory of Heredity.* By A. SEDGWICK, M.A.

(2) *Suggestions with regard to the nervous system of the Chordata.* By W. BATESON, B.A., St John's.

Having in view the facts of the anatomy and development of the Enteropneusta, and especially the condition of the delaminated portion of nervous system which is connected with the skin by median-dorsal chords, the author argued that

(1) The nervous system of the chordata must have been from the first an unpaired and "unsegmented" structure.

(2) That its present origin by invagination must be secondary to a primitive process of delamination.

(3) That the difference of function between the dorsal and ventral roots must have been primitive, the result of the physical necessities of the case.

If these suggestions be accepted, the nervous systems of the lower chordata form a regular and progressive series.

(3) *On the nature of the Heart-Sounds.* By F. J. ALLEN, M.A., St John's.

The difference between the two sounds of the heart may be one of degree rather than of nature. Certain circumstances render the sounds more alike, and certain others increase their difference.

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There is no real similarity between the systolic sound and the susurrus of muscles in tetanic contractions: the former is quite smooth, the latter rough. A prevailing fundamental note may be heard in each: but this is also heard in the *second* sound; and is therefore independent of muscle, and may be the fundamental note of the observer's auditory mechanism.

In the author's more delicate experiments a stethoscope of new form was used. This was an ordinary binaural flexible stethoscope, fitted with a peculiar receiver which consisted of a small drum with heads of thin caoutchouc membrane. This instrument, called (in default of a better name) the tympanoid stethoscope, is very sensitive; since the vibrations of a body can be communicated to it by a very light contact with the outer drum-membrane.

Various kinds of muscle were examined both within and outside the body. It was found that skeletal muscle, both of the frog and of mammals, gave rise to an audible concussion or thud, when contracting under the stimulus of single induction shocks. This thud is louder when the muscle pulls on a weight; but nevertheless it appears to be independent of resistance: for it is still produced when the muscle lies loose upon the membrane of the stethoscope, and is even audible when a scrap of muscle no bigger than a pea is used. Little or no sound is produced, however, when the muscle is exhausted, even though its contraction remains visible to the eye.

The heart-muscle of warm-blooded animals, stimulated in the same way, produces a similar concussion or thud; which, however, is much duller and weaker than that of the skeletal muscles. The heart muscle-sound has not the character of a susurrus, but appears to be merely the expression of a single contraction less sudden and of longer duration than that of the skeletal muscle.

The heart-muscle of cold-blooded animals (frog and crocodile) gives rise to no sound audible with the tympanoid stethoscope. This is apparently because the contraction, although of much longer duration than those before mentioned, is not sudden enough to produce an audible concussion.

It appears that in the following kinds of muscles there is an ascending scale of suddenness in contraction:—(1) Unstriped muscle, (2) cold-blooded heart-muscle, (3) warm-blooded heart-muscle, (4) cold-blooded skeletal muscle, (5) warm-blooded skeletal muscle. The first two contract too slowly to give rise to an audible concussion, and the last produces a very sharp concussion.

Making experiments with the tension sounds produced by suddenly stretching or jerking membranous substances, such as tape and leather, I find that a sudden and weak jerk gives rise to a short flapping sound, whilst a slower jerk with sustained

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tension produces a long musical sound. The relation between the two kinds of sound is very suggestive of the relation between the second and first heart-sounds.

Since the cardiac muscle can produce a contraction sound, it is evident that it must contribute a share to the first heart-sound, although it may not be an important factor. If it were important, the sound of a hypertrophied heart should be loud; whereas experience shews the opposite to be the case. The sound of the muscle is probably not a prolonged one, but simply a dull thud: and, since the muscle must contract before any other systolic event can take place, *the muscular thud must come at the beginning of the systolic sound*, not prolonging it towards the end as usually supposed. The booming prolongation of the sound is most likely due to the vibration of the valves and chordae tendineae under the sustained tension to which they are subjected; as simulated in the experiments on stretching membranes.

The systolic sound in small animals (cat, rat, mouse) is not prolonged as in man and other larger animals. The two sounds in small animals are nearly alike. The conditions which give the special characters to the systolic sound in the human and other large hearts, appear to be (1) the long systole with sustained tension, (2) the large size and consequent free vibratility of the valves and chordae tendineae, and (3) the thickness of the ventricle-wall, which is not favourable to sharp vibration in itself, and which acts as a partial non-conductor to the valvular sound. The similarity between the two sounds in small hearts should therefore be owing to the absence of these special conditions.

(4) *On the travelling of the Transpiration Current in the Crassulaceæ.* By F. W. OLIVER, Trinity College.

In this communication the author drew attention to some experiments carried out by him on certain members of the order, which would point to the importance of the living cells of the wood in the ascent of the sap. The cooperation of the living elements has been already shewn to be logically necessary by the physiologists Godlewski and Westermaier; and from the experiments described some support for their general theories was attempted to be drawn for a special case.

(5) *On the constitution of the walls of vegetable cells and the degeneration changes occurring in them.* By WALTER GARDINER, M.A., Clare College.

The author stated that although a wall consisting entirely of unaltered cellulose may conveniently be regarded as the typical cell-wall, yet practically such a structure is seldom met with in

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vegetable tissue. Even from the very first the wall may be for the most part mucilaginous, or may consist of some form of hydrated cellulose; but perhaps the more normal phenomenon that occurs is, that in a thickening cell-wall the innermost layers consist of cellulose, while the outer portion tends to degenerate into mucilage.

The structure known as the middle-lamella is composed, in the pure cellulose cell-wall, simply of cellulose layers, which have become peculiarly dense and resistant owing to pressure and tension, but more generally these layers undergo a hydrated or mucilaginous change, and even incipient lignification and cuticularisation.

The occurrence of mucilaginous degeneration is one of the deepest physiological interest, since by this means the separation of cells in the formation of stomata, intercellular spaces, and the like is made possible. During the separation from one another of contiguous cells the external mucilaginous portions of the wall frequently present a very rugged outline, and the two walls may even be connected by strands of mucilage which traverse the intercellular space.

The external portion of the walls lining intercellular spaces often degenerates into mucilage over their whole surface, but in many cases this degeneration is confined to certain definite areas, leading to the formation of mucilage rods and drops.

The process of partial or entire mucilaginous degeneration appears to be one of hydration, since (among other things) it can be shewn that by employing suitable hydrating reagents, the area of mucilaginous change can be increased at will. The author believes that the phenomenon of swelling as exhibited by cell-walls is made possible by, and is in the main dependent on, the formation of definite hydrates of cellulose, such hydrated forms being very unstable and probably consisting of a large number of molecules.

In addition to the mucilaginous degeneration occurring in the walls of normal cells, mucilage may be derived directly from the protoplasm or from walls which are undergoing a pathological change.

Cuticularisation occurs in the outermost layers of the walls of living cells, when the walls in question are freely exposed to the external medium, but in the opinion of the author, lignification and suberisation is in some way dependent on changes taking place in consequence of the slow and gradual death of the cell, which must however be situated in the vicinity of living tissue. In certain instances a suberous degeneration may occur in the most external layers of certain cellulose walls, in which normally a mucilaginous change would have supervened, but this is apparently dependent

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on some pathological change. Instances occur also in which cell-walls of a cellulose nature when living give a definite lignin reaction when dead. In lignified and suberised walls the maximum change occurs in the neighbourhood of the middle lamella, and here too the change is associated, although to a less degree, with mucilaginous degeneration.

In conclusion the author stated that he had undertaken the above investigation mainly for the purpose of ascertaining whether intercellular protoplasm was ever present, in plant tissue. All his frequently repeated observations most emphatically negatived the occurrence of any such substance.

November 23, 1885.

PROFESSOR FOSTER, PRESIDENT, IN THE CHAIR.

Mr F. J. Allen, M.A., St John's College, was elected a Fellow.

The following communications were made to the Society :—

(1) *On a new method of producing the fringes of interference.* By L. R. WILBERFORCE, B.A., Trinity College.

[*Abstract.*]

The author stated that in the course of an inquiry into the suitability of various forms of interference-fringes for certain investigations on the velocity of light upon which he had been engaged, he had been led to adopt the mode of production which was the subject of his paper.

He briefly described the method, indicated the elements of its theory, and, by a comparison of his results with those of former experimenters, shewed the great increase of accuracy attainable by means of it.

The paper is being printed in the *Transactions* of the Society.

(2) *Some experiments on the dielectric strength of mixtures of gases.* By Dr C. OLEARSKI. Communicated by Prof. J. J. THOMSON.

G. Wiedemann and Rühlmann in their investigations on the passage of electricity through gases[1] compared the electric strength of oxygen, nitrogen and air free from carbonic acid and aqueous

[1] *Pogg. Ann.* Vol. 145, S. 235, or Wiedemann *Elektricität*, Vol. IV. p. 460.

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vapour. They determined the quantity of electricity passing at each discharge, and, although it is perhaps not quite evident, it is at least very probable that the quantity of electricity carried at a discharge is proportional to the potential necessary to introduce the discharge. Based on this supposition we should conclude from the diagram[1] given by Wiedemann and Rühlmann that there is an anomaly as regards the strength of air, which according to them would be intermediate between its constituent gases for pressures larger than 30 mm. of mercury, but weaker than either nitrogen or oxygen for pressures lower than that. This would not be what we could easily explain, and would suggest that for these low pressures only the weaker constituent of the mixture is carrying the discharge.

But as this result does not seem to be conclusively proved by the experiments quoted above, and is not even announced by the investigators, and it was of some interest to make experiments on the strength of mixtures of gases, Prof. J. Thomson suggested to me during my stay in Cambridge to investigate the question by the method of Faraday[2]. Accordingly a current given by a Ruhmkorff coil was sent through a multiple conductor, in one branch of which was a glass tube (whose internal diameter was 2 cm.), closed air-tight by india-rubber stoppers, having two spherical platinum electrodes at a distance of about 15 cm., while in the other branch there was the usual apparatus for measuring the length of sparks, consisting of two movable brass balls in open air.

When the distance of movable electrodes is large enough, sparks pass only in the glass tube, when it diminishes they appear between the movable spheres as well. The limit distance measures the strength of the gas enclosed in the tube. There is however a difficulty in making an exact measurement, for, when the distance of electrodes is reached at which the first spark passes across them, the discharge does not always choose the same way. In this manner there is an interval which is larger for higher pressures than for lower ones, in which it is impossible to decide whether the gas enclosed in the tube or the air between the movable electrodes is stronger. But when we only compare the strengths of different gases, faults coming from this incertitude may be possibly eliminated by keeping always the same mode of proceeding. For every determination therefore the distance of the movable electrodes was at first made so small that all discharges passed through them. Then the gas was pumped out by means of a Sprengel pump and the pressure was noted at which the first dis-

[1] l. c. tab. IV. fig. 2, or Wied. *Elektr.* Vol. IV. p. 462, fig. 193.
[2] *Experim. Research.* Vol. I.

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charge passed in the tube. This mode of investigation was used in all the experiments, whose results are given below.

There is another point which may be noticed. Faraday originally observed that after a discharge is passed across a gas it becomes electrically weaker than it was before. It may be easily shewn that this alteration in the electric strength of the gas lasts for some time. For instance I made the distance between the movable electrodes 0·52 mm., and having filled the tube with nitrogen I found that until it was pumped out to 10 mm. of mercury no discharge appeared in the tube. After having introduced a little more nitrogen, I passed the discharges from a Ruhmkorff coil through the tube for 10 minutes; after waiting for 2 minutes, I found the corresponding pressure of nitrogen 12·5 mm., which shews that nitrogen is weakened by the passage of discharges, and does not acquire its previous properties after a rest of 2 minutes. To avoid errors arising from this cause, after every determination some new gas was introduced into the tube (when the gas was very rarefied sufficient to increase its pressure about 30—40 mm.). In this manner the gas enclosed in the tube was never sensibly changed by previous discharges.

Experiments made at different times have not given exactly concordant results in absolute numbers for the same gases, nevertheless the order of gases as regards their electric strength was always the same and the ratio of numbers found for several gases was nearly equal. Even for absolute numbers the accordance was much better at low pressures when the discharge had the form of a glow, than at higher ones when the passage of electricity is accompanied by a spark. These differences are doubtless principally caused by the different states of free atmospheric air, by the varying quantity of dust in it, and perhaps by some differences in the working of the Ruhmkorff coil. On the contrary, measurements made some hours one after another gave numbers which never disagreed 10% even for pressures larger than 100 mm. of mercury. Therefore finally I made experiments with gases, whose electric strength I compared, on the same day, and I shall give below only such measurements or means of such measurements which were executed with this caution.

The following is an example of measurements executed one after another with oxygen, a mixture containing 61% vol. oxygen and 39% vol. nitrogen, air and nitrogen under low pressures. Oxygen which was prepared from chlorate of potash was led through water and a solution of caustic potash. The carbonic acid contained in the air was also removed by means of caustic potash. Finally the gases were dried by passing through two tubes with $CaCl_2$ and the third with phosphoric anhydride, each of which contained a plug of wool to retain the dust.

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Distance of movable electrodes.	Pressure in mm. of mercury. Oxygen.	Mixture of Oxygen and Nitrogen 61 °/₀ vol. 39 °/₀ vol.	Air.	Nitrogen.
27·7 mm.	135	140	133	
18 „	101	100	98	
12·6 „	87	90	84	
3·9 „	73	71	72	74
1·92 „	52	53	53	55
1·2 „	27	25	28	30
0·7 „	14	12·5	16	15
0·52 „	9·5	8·5	11	12
0·35 „	5·5	5·0	6·	6·5

In each horizontal line the distance between the movable electrodes is given and the corresponding pressures of different gases at which the first discharge appeared in the tube.

From these experiments air would seem to be intermediate between its constituent gases; while the mixture of oxygen and nitrogen in the ratio of 61°/₀ vol. to 39°/₀, is stronger even than oxygen. The difference is however not larger than the possible errors of the experiment. Another experiment has given for the same mixture and for oxygen numbers almost exactly equal.

The following are the means of three measurements every one of which included oxygen, air and nitrogen:

There is a very small difference in the electric strength of oxygen, nitrogen and air. Oxygen is a little stronger than nitrogen for low pressures just as Wiedemann and Rühlmann found, whereas Faraday has shewn that under the pressure of an atmosphere nitrogen is the stronger. Generally the curves given on the diagram of Wiedemann and Rühlmann agree with numbers found here as measuring the electric strength, which shews that quantities of electricity carried at a discharge measured by Wiedemann and Rühlmann are proportional to the potential which is necessary to introduce the discharge. Air appears however to be intermediate between oxygen and nitrogen even under the pressure of 6 mm. of mercury.

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Distance of movable electrodes.	Pressure in mm. of mercury.		
	Nitrogen.	Air.	Oxygen.
27·7 mm.	126	122	126
18 „	102	108	105
12·6 „	90	95	94
3·9 „	73	68	68
1·92 „	50	44	46
1·2 „	30	24	25
0·7 „	15	13·5	13
0·52 „	11	9·5	9·5
0·35 „	6·5	5·5	5·7

Hydrogen is very much weaker than nitrogen, and therefore I compared it with a mixture containing 70% vol. hydrogen and 30% vol. nitrogen.

Distance of movable electrodes.	Pressure in mm. of mercury.			
	H.	N.	Mixture observed.	70% H. 30% N. Calculated.
27·7 mm.	240	140	166	199
18 „	210	113	135	170
12·6 „	180	99	122	147
3·9 „	123	78	102	106
1·92 „	76	48	66	66
1·2 „	51	29	44	42
0·7 „	31	16	23	24
0·52 „	21	12	14	17
0·35 „	9	6·5	7	8

In the last column are given pressures calculated when it is

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supposed that the strength of a mixture is equal to the sum of strengths of its constituents, i.e. according to the formula $M = \frac{H.N}{0{\cdot}7N + 0{\cdot}3H}$, where M is the pressure of the mixture, H and N pressures of hydrogen and nitrogen. It seems that calculated and observed pressures disagree more than it could be explained by the imperfection of the proportionality of strength and pressure of gases and by errors of experiments.

(3) *On the mutual action of oscillatory twists in a vibrating medium.* By A. H. LEAHY, M.A.

This paper is being printed in the *Transactions* of the Society.

(4) *On the transpiration-stream in cut branches.* By FRANCIS DARWIN, M.A., Trinity College, and REGINALD W. PHILLIPS, B.A., St John's College.

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§ 1. *Object of the Experiments.*

The series of experiments here recorded was commenced in the spring of 1884, and was in large part completed in that year. We have been forestalled in some of our results since the work was begun, nevertheless we think it best to publish them, since the amount of evidence available on the subject is by no means large.

It is not necessary to refer in detail to the views held by different botanists as to the way in which the transpiration-stream passes through wood. The subject has been recently discussed in an admirable paper by Elfving†, and more recently in Godlewski's‡ paper on the same subject.

* i. e. Branches sawn half through on opposite sides at points near each other.

† Ueber den Transpirationstrom in den Pflanzen. *Acta Soc. Scient. Fennicæ.* Tom. XIV, 1884.

‡ Pringsheim's *Jahrbuch.* 1884. Bd. XV, Heft 4.

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It will suffice to say that the two chief theories are

i. That the chief and essential path of the stream is in the walls of the elements of the wood; that the stream is in fact the movement of the water of imbibition contained in the walls of the elements.

This view we shall refer to as the "imbibition theory".

ii. That the chief and essential path for the passage of the stream is *in the cavities* of the elements of the wood.

The chief object of our experiments has been to help to decide the question which of these theories is the right one.

In the present paper we have not attempted to do more than to test the validity of the two theories as applied to *cut* branches, without considering whether or no the conclusions arrived at are applicable to rooted plants. The method employed may, it is believed, be used with plants grown in water.

§ 2. *Method.*

The experiments here recorded were chiefly made with the potometer already briefly described by one of us*. The instrument is intended to measure the amount of water absorbed by a transpiring plant in a given time, and has been called the potometer in imitation of the name suggested by Moll† for a different instrument.

There is nothing new in the principle of our potometer, the chief merit that we may claim for it is its convenience for working purposes.

Fig. 1 shows the potometer in action. It consists of a T tube, of which the limb *a* is bent so as to be parallel to the remaining two limbs, and into which a cut branch of a plant is fixed by an india-rubber tube. The two other limbs of the tube are closed by india-rubber corks, through one of which passes a thermometer tube *b*. The T tube and the thermometer tube being filled with water the apparatus is fixed so that the end of *b* dips into a small vessel of water, *c*. It is obvious that all the water which the branch absorbs must be obtained from the vessel *c*. To take a reading with the instrument all that is necessary is to remove the block of wood *d* so that the vessel *c* can be lowered; when this is done air instead of water will be sucked in at the end of *b*. When a

* Francis Darwin in *Nature*, May 1, 1884.
† *Archives Neerlandaises* xviii. 1884.

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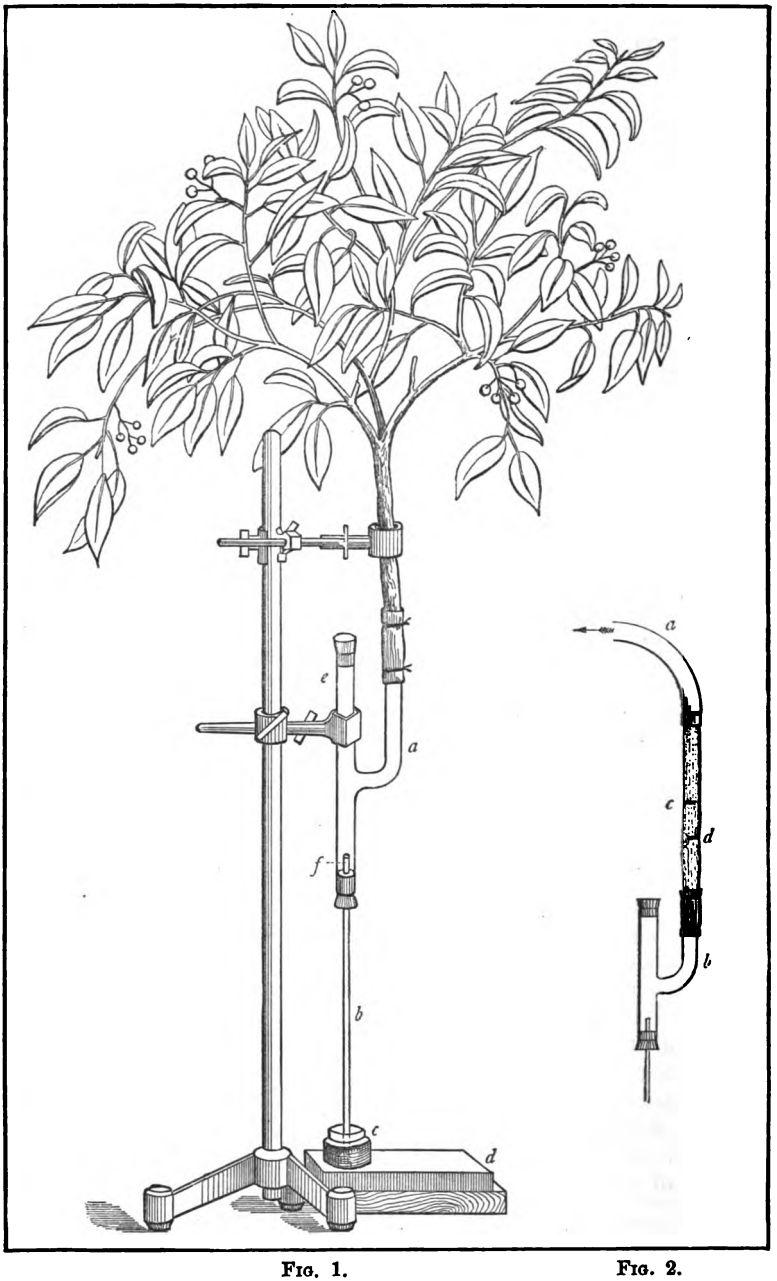

FIG. 1. FIG. 2.

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column of air a few millimeters in length has entered the tube *b*, the vessel *c* is replaced; by this means a bubble of air is enclosed in *b*; the bubble travels up the tube and serves to indicate the rapidity of the current of water in the tube *b*. The time which the bubble takes to traverse a measured length of tube is read by means of a stop-watch. Then by taking the reciprocals of these readings a series of figures proportional to the amounts of water absorbed by the branch in a given time are obtained. Thus if 10″ is the reading, the reciprocal being 0·1, the rate of absorption is put down as 100; five seconds being entered as 200; 20″ as 50, and so on. The actual quantities of water corresponding to these figures vary according to the size of the tube used. A rate of 100 in our experiments means that some quantity between 4 and 8 gramms of water are absorbed per hour.

Details. It will be seen that at each reading a small bubble enters the potometer, these bubbles collect at *e*, and can be got rid of at intervals by taking out the cork *e* and filling up the vacant space with water. In some rare cases bubbles have been found to collect in the limb *a* underneath the cut end of the branch, this is of course a serious error, and to avoid the possibility of its occurrence, a different form of instrument was occasionally used, in which any bubbles entering from the end of the branch could be collected with the bubbles used as indices of the rate of absorption. Practically however it was found that the form given in fig. 1 did not lead to errors from this cause.

There is naturally some resistance to the drawing of the bubble of air into the end of *b* (fig. 1); thus it follows that the bubble enters with a jerk, and does not settle down to a steady pace until it has travelled some way up the tube. It is therefore necessary that the mark on the tube forming the lower limit of the fixed distance to be traversed by the travelling air-bubble should be at some little distance from the free end of the tube; in our experiments the length of *bc* was usually about 10 c. m. It was found convenient to employ the upper end (*f*) of the thermometer tube as the upper limit of the measured space.

In using the apparatus it is of importance that the length of the bubble employed as an index should be constant, since it was found that by using long and short bubbles alternately the readings could be made to vary; the longer bubbles giving slightly quicker rates than the shorter ones. This point was carefully attended to, each bubble being regulated to a measured length as it entered. With these precautions fairly uniform readings can be taken. The following series is not specially selected, but shows the kind of uniformity easily attained to—and greater uniformity was often reached:

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5·6 seconds
5·8 „
6·0 „
5·8 „
6·0 „
5·8 „
5·8 „

In our experiments we usually employed Cambridge water-supply water, which is fairly free from impurities, in some cases however distilled water was used. The success of the method depends on there being no leaks in the apparatus; it is therefore essential that the india-rubber corks should be good, and the ligatures where the india-rubber tube is attached either to the glass or the branch should be carefully made. For this purpose either copper wire or (what is better) india-rubber thread was employed.

Advantages and disadvantages of the method.

The apparatus is rapidly put together so that readings can be taken within one minute after the branch has been cut. It is so simple that it is not likely to get out of order, and the places of possible leaks are reduced to the lowest possible number. Each reading takes only a few seconds, so that a number of observations may be made in a short space of time. It is easy to take observations without in any way disturbing the plant, and this is of importance since it has been shown that the shaking caused by such a disturbance as is involved in weighing a transpiring plant influences the transpiration*.

Owing to the sinking of the level in the vessels *c* and *e*, as the branch absorbs water, the conditions do not remain absolutely identical from hour to hour, but since these sources of error are extremely small, and their effect is spread over a considerable interval, they may be considered as of no moment.

Changes in the temperature of the water, or in the pressure of the surrounding air, must theoretically disturb the reading of the instrument. If the water in the Y tube is increasing in bulk owing to increase in temperature the passage of water up the tube *b* must be delayed.

If it be remembered how slowly the temperature of the 17 cc. contained in the potometer would be changed in the course of experiments conducted in ordinary rooms, and when it is further considered that the bubble serving as index often moves at the rate of 12 mm. per second, it will be seen that no serious error

* Baranetzky. *Bot. Zeitung*, 1872.

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will be introduced. The same considerations *mutatis mutandis* apply to barometric changes.

It must further be remembered that the use to which we have put the potometer is especially to *test the effect of sudden changes* in the conditions; and that where we record changes in rate of absorption, which are observable from minute to minute, errors of this sort need not be considered.

Tests applied to the apparatus.

We were not able to discover any serious sources of error in the apparatus, but we were anxious to prove that the readings of the stop-watch do represent the flow through the potometer. We therefore substituted a siphon at *a* in place of the plant, and proceeded to compare the different rates of flow of water through the siphon with the corresponding readings of the potometer. It was found that low pressures of the siphon corresponded to what we were accustomed to consider high readings of the potometer. And as we further found that the liability to error is greater with rapidly moving bubbles, we may assume that our experimental readings are *more* and not *less* trustworthy than those obtained in the testing experiments.

The siphon was so arranged that the free end could be raised and lowered, and the experiment here given begins with a pressure of 100 mm. of water:—that is to say, the efficient column of water was 100 mm. in height; as the column was increased by regular increments to 200 mm., the reading of the stop-watch decreased from 7·3″ to 3·5″.

In the following table *A* gives the reading of the stop-watch; *B* gives a series of figures proportional to the reciprocals of the figures in *A*; *C* gives figures proportional to the pressure of the siphon,—figures which therefore approximately represent the outflow from the siphon:

A	*B*	*C*
7·3″	100	100
5·8	125	120
5·0	146	140
4·5	162	160
4·0	183	180
3·5	209	200

It will be seen that the difference between *B* and *C* never reaches 5°/₀.

With still higher pressures and with a correspondingly rapid rate in the passage of the bubble the error increases from 15 to

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20°/₀, so that we are not inclined to trust observations where the reading of the stop-watch is less than 3″.

A second method of testing the apparatus was that of weighing the quantity of water flowing from the siphon in given times; the siphon was allowed to run for 15 minutes, the potometer being read at intervals of about a minute. The current was then quickened and the amount of water yielded in 15 m. again weighed, the potometer readings having been again taken.

In the following table, column *A* gives numbers proportional to reciprocals of the mean of the reading of the stop-watch taken during the flow of the siphon at one pressure. Those in *B* are proportional to the weights of water flowing from the siphon in a quarter of an hour at three different pressures. (The actual pressures are not given.)

A	*B*	error
100	100	„
146	150	2·7°/₀
408	430	4·4°/₀

If we call *B* the actual outflow from the siphon, and *A* the outflow estimated by the potometer, we shall see that the error does not reach 5°/₀. The reading of the stop-watch corresponding to the amount 408 in column *A* was 5·2″. If the calculated flow had been equal to the observed flow the reading would have been 4·9″, and since in our experiments we should not base any conclusions on a difference of 0·3″, it is clear that the amount of discrepancy between the observed and the calculated outflow is not of an amount to influence the trustworthiness of our conclusions.

But the best proof of the trustworthiness of the apparatus was obtained by comparing the readings of the potometer, obtained from a transpiring branch placed under varying conditions of atmospheric humidity, with the readings of a psychrometer. It is known that the amount of water absorbed by a cut branch varies inversely with the variations in the relative humidity of the atmosphere when the temperature remains constant*. It is clear therefore that the readings of the potometer should bear a certain ratio to those of the wet and dry bulb thermometer. The plant used was a branch of Portugal laurel, the twigs being somewhat altered in position by tying so as to give the whole a convenient form for being placed under a bell-jar. The branch was passed through a cork fitting into a horizontal plate of glass on which the bell-glass could be placed so that the plant and the thermometer could be exposed to a moist atmosphere, the humidity

* C. Eder, "Untersuchungen über die Ausscheidung von Wasserdampf bei den Pflanzen." *Sitz. der k. Akad. der Wiss.* 3. Wien, Bd. 72, Oct. 1875.

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of which could be varied by raising one edge of the jar. When the results calculated from the potometer and psychrometer readings, under these circumstances, were represented graphically, it was found that the two curves were strikingly similar.

Precautions.

Sachs* has pointed out that the amount of water absorbed by cut branches suffers a gradual diminution so that long-continued observations are not trustworthy. But since our results chiefly depend on sudden changes in the rates of absorption, changes which are made evident in a few minutes, we are not inclined to believe that this diminution introduces any serious error into our results. For similar reasons we have not thought it necessary to give the readings of the psychrometer which were noted in nearly all cases.

Another phenomenon which occurs when a cut branch is placed in the potometer must be noticed, since, unless regard is had to it in experimenting, serious errors must arise. When a branch (e. g. of Portugal laurel) is cut, and fitted into the potometer, the readings are at first very high but rapidly sink, until after an hour or so they reach approximate constancy. The following is an example:

Sept. 16, 1884. Portugal laurel (*Prunus lusitanica*) cut under water and at once fixed in the potometer:

p. m.	rate
3·37...	...263
43...	...208
50...	...167
54...	...159
4·13...	...118
5·3 ...	... 87
37...	... 76
41...	... 80

During this experiment the air was becoming somewhat damper, which would slightly increase the fall in the rate of absorption.

§ 3. *On Dufour's experiments.*

The rival theories of water transport, i. e. the imbibition and the "intracavital" theory, have been tested by Dufour in a paper published in Sach's *Arbeiten*, 1884. If Dufour is right, the intra-

* *Flora*, 1856.

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cavital theory of water-transmission must be condemned. It is therefore of importance to test the validity of his results. The Essay consists of several sections—we shall first deal experimentally with one of these.

Dufour made a series of experiments in which two incisions were made on two opposite sides of a branch and at a short distance apart, as shown in fig. 2 or 7. If the incisions reach the centre of the branch the continuity of all vessels must be broken. Branches thus treated were tested as regards their power of transmitting the water of transpiration, and also as to their power of transmitting a current of water forced in under pressure.

He was able to show that a transpiring branch which had been "doubly sawn*" can transmit a sufficient amount of water to prevent the withering of the leaves; but that when the branch is cut off above and below the double saw-cuts and it is attempted to force water through the branch it is found that the current cannot pass the double cuts.

From this it is argued that the transpiration-stream which *can* pass the double cuts must be of a different nature to the filtration-stream (the stream travelling in the cavities of the elements under pressure) which cannot pass this part of the branch. That is to say, that the argument leads up to the belief that the stream of water travelling in a transpiring branch does not travel in the cavities of the wood elements. That it travels as water of imbibition in the cell-walls seems the only other theory which suggests itself.

We believe that Dufour is wrong in two points:

(i) He has not realised how great is the obstruction to the transpiration-current produced by double-sawing.

(ii) He has exaggerated the difficulty of forcing water through a doubly-sawn branch. We believe when the transpiration-stream and the filtration-stream are properly estimated, that they will be seen to be equally transmissible through a doubly-sawn branch.

We proceed to the experimental proof of these statements.

§ 4. I. *Estimation of the transpiration-stream in doubly-sawn branches.*

In the paper already referred to† it has been shown that in the case of Portugal laurel, double-sawing may produce a great effect. The fall in the rate of absorption was from 100 to 3·2‡.

* i.e. sawn as shown in figs 2 and 7.

† Francis Darwin in *Nature*, May 1, 1884.

‡ The cuts were ½ inch apart.

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Kohl has afterwards made a similar experiment with Pinus*. The following series are of the same character:

EXP. 1. July 23, 1884. Portugal laurel (*Prunus lusitanica*).

	Time a.m.	Rate
	11·40	81
	12·2	83
	6	83
	11	85
(i)	13	Sawed half through†
	14	68
	16	70
	19	76
	25	76
(ii)	27	Sawed opposite to (i) and 2 cm. above it
	39	7
	1h 0m	10
	3·23	27

This experiment shows two points.

(i) A single cut produces no very great diminution in the rate of absorption, whereas the second cut produces a great fall.

(ii) After the first cut the diminished rate of absorption begins at once to rise. The great diminution due to the second saw is also temporary, and a slow recovery takes place.

EXP. 2. August 1, 1884. Portugal laurel.

	Time p.m.	Rate
	4·25	53
	4·31	53
(i)	Sawed half through	
	4·33	47
	39	48
(ii)	40	Sawed opposite to (i) and 2 cm. above it
	41	11
	57	13
	5·20	16
	6·9	22

* *Bot. Zeitung*, 1885.

† The sawings were in all cases made at a fair distance, i.e. 10 to 15 cm. from the cut end of the branch.

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It will be seen that the amount of effect in Exp. 2 is not so great as in Exp. 1. The variability is due as we believe chiefly to the difficulty of making the two saw-incisions exactly opposite. In the following experiment the incisions were carefully made, and we believed at the time that the whole of the vessels in the cross section were severed.

Exp. 3. July 27, 1885.

	p.m.	Rate
	5·26	84·7
(i)	29	Sawed half through
	31	80·0
	48	78·1
(ii)	52	Sawed half through opposite to (i)
	54	29·4

It was found afterwards upon careful examination that the saw-cuts were not however exactly opposite, and that therefore a very small strip of tissue remained unsevered. This no doubt accounted for the somewhat slight effect produced by double sawing.

We therefore determined to make the incisions so deep as to preclude this kind of error. The following experiment shows the great effect of slightly deepening the saw-cuts.

Exp. 4. July 31, 1885. Portugal laurel.

	Time a.m.	Rate
	10·58	97·1
(i)	Sawed half through	
	11·5½	78·1
	7	86·9
(ii)	10	Sawed half through opposite to (i)
	13	56·2
	16	59·9

The incisions* seemed only just to include the whole section and certainly did not overlap. They were therefore both deepened to 5 mm. (the branch being 8 mm. in diameter). The rate of absorption immediately fell enormously.

a.m.	Rate
11·22	6·4

* Incisions 5·5 cm. and 7·5 cm. respectively from the cut surface.

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The following experiment gives similar results:

EXP. 5. Portugal laurel cut August 11, 1885.

Aug. 12.	Time	Rate
	12·29	82·0
	43	83·3
	1·4	82·0

(i) Sawed through $\frac{7}{13}$ of diameter (13 mm.) at 13 cm. from the cut end:

1·13	76·9
15	76·9

(ii) Sawed 4 mm. deep opposite to, and 2 cm. above (i):

1·18	68·5
20	72·5

(iii) Deepened (ii) to 6 mm., i.e. $\frac{6}{13}$ of diameter:

1·22	17·9
26	19·0

(iv) Deepened both (i) and (iii) to 8 mm., $\frac{8}{13}$ diameter:

1·33	11·0

The following examples show that when the two incisions undoubtedly overlap, the fall in rate after the second incision is great:

Portugal laurel cut from the tree on July 28, 1885.

EXP. 6. July 31.

a.m.	Rate
11·57	166·7
59	156·3
12·9	158·7

(i) Sawed through $\frac{8}{12}$ of diameter*:

a.m.	Rate
12·13	90·9
15	95·2

(ii) Sawed through $\frac{6}{12}$ of diameter opposite (i):

12·23	6·9
37	8·1

EXP. 7. Portugal laurel cut from the tree August 11, 1885.

August 12.	p.m.	Rate
	12·48	90·9

* Incisions at 8 and 10 cm. from the cut surface.

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Here the first cut diminished the rate by about 30°/₀; the immediate effect of the second cut being a diminution of 77°/₀. Partial recovery occurred after the first sawing, but no signs of recovery were seen after the second.

In a plant such as the Jerusalem Artichoke (*Helianthus tuberosus*) where the vessels take a larger share in the vascular bundle than they do in woody plants, the effect of the two cuts is greater still, that of the second cut being very strongly marked.

Exp. 13. Aug. 15, 1884. Jerusalem Artichokes (cut under water).

p.m.	Rate
5·49	143
55	128
58	128
6·1	Cut half through
3	57
7	58
9	58
10	Cut half through opposite, 2 cm. above
6·30	2 (about)

The reading after the second cut was difficult to take, the bubble took about 10 minutes instead of 17·2″. But whether or not it is accurately represented in the column of reciprocals by the figure 2, there is no doubt that the effect of the second cut was enormous as compared with the first.

The same thing is shown in the following experiment:

Exp. 14. Aug. 15, 1884. Jerusalem Artichoke (cut under water).

p.m.	Rate
4·55	116
5· 3	116
4	Cut half through
5	91
6	98
9	104
17	104
18	Cut opposite, 2 cm. above
about 30	2 (about)

The effect of the first cut is small, and is followed by partial recovery, while the second cut produces a comparatively enormous

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It is interesting to compare the results in the last two experiments with the effect of double cuts on gymnosperms.

Exp. 15. Aug. 14, 1884. Scotch fir (*Pinus sylvestris*), (cut under water).

	a.m.	Rate
	11· 0	53
	5	52
	9	50
	17	49
	22	48
	27	47
	29	47
(i)	32	Sawed half through
	34	45
	35	48
	39	47
	45	47
	54	47
	p.m.	
	12· 1	49
(ii)	2	Sawed opposite to (i) and 2 cm. higher
	5	45
	7	45
	12	46
	19	46

Here neither the effect of first nor second cuts is at all marked, but in another experiment the effect was greater.

Exp. 16. *Pinus sylvestris*, cut under water.

	Time	Rate
	3·3	64
(i)	5	Sawed through $\frac{9}{16}$, of diameter*
	7	60
	19	67
(ii)	21	Sawed through $\frac{9}{16}$, opposite to and 2 cm. above (i)
	28	34

In two other experiments the effect of doubling sawing was to cause a depression represented by 100 to 50, and 100 to 33.

With Yew (*Taxus baccata*) the effect is also somewhat variable.

* Diameter of branch 16 mm., incisions 13 and 15 cm. from cut end.

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EXP. 17. Yew. Aug. 11, 1884:

	a.m.	Rate
	11·25	177
	p.m.	
	12·12	195
(i)	14	Sawed more than half through
	15	190
	19	189
	32	195
	1· 0	197
	11	204
	17	209
(ii)	21	Sawed opposite to (i) $1\frac{1}{2}$ cm. above it
	24	152
	28	167
	31	153
	36	172
	3·30	187

Here the first cut produces a small effect, the second cut a greater one. The experiment is somewhat injured by the fact of the amount of absorption being on the increase during the experiment.

EXP. 18. Yew. Cut from the tree Aug. 17, 1885.

Aug. 18.	p.m.	Rate
	3·43	21·0
	45	21·0
(i)	48	Cut through $\frac{9}{16}$ of diameter
	49	21·0
	57	21·6
(ii)	58	Cut through $\frac{9}{16}$ of diameter* opposite (i) and 2 cm. higher
	4· 0	20·6
	7	21·3

Here (in Exp. 18) the double sawing produced practically no diminution in the rate of absorption.

EXP. 19.

Aug. 18, 1885.	p.m.	Rate
	12·37	40·0
	54	36·5
(i)	55	Cut through $\frac{2}{3}$ of diameter
	57	36·0
	1· 3	35·2
(ii)	5	Cut through $\frac{1}{2}$ the diameter opposite (i) and 2 cm. higher up
	7	33·3
	27	30·9

* Diameter of branch 16 mm., incisions 13 and 15 cm. from cut end.

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If the result be graphically represented, it will be seen that the incisions make practically no difference in the gradual fall perceptible in the rate of absorption.

In another experiment the results were as follows:

The first cut produced no slowing, but a slight quickening of the rate. The second cut produced a slowing represented by 100 to 91·5.

Lastly, in another experiment the amount of slowing was 100 to 96.

The following is an abstract of the above potometer experiments. The figures given represent the amount of slowing produced by double sawing. The rate before sawing being taken as 100*.

Portugal laurel	100 to 8·2	Pinus sylvestris.....	100 to 96
	20·8		50·8
	6·2		50
	13·4		33
	4·4	Taxus baccata	72·8
	6·6		95·5
Beech	28·6		89·2
Ivy	15·9		91·5
Sycamore	11·7		96·0
	33·0		
Elder	5·0		
H. Chestnut	38·0		
	18·3		
Helianthus tuberosus ...	1·6		
	1·7		
Average	100 to 14·2	Average	100 to 74·9
Average of trees excluding Helianthus	100 to 16·1		

The different results obtained, in most cases, with Gymnosperms as compared with Angiosperms will be discussed later on.

Effects of double sawing estimated by weights of water absorbed.

The following experiments were made with the view of confirming by another method the experiments made with the potometer. The branches were fixed with their cut ends in water, and the amounts of water absorbed in a given time were estimated by weighing the vessels at regular intervals. The branches were then

* Where the rate increased after the first cut owing to the air becoming drier, &c. &c., the rate *after* the first cut is taken as 100.

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sawn half through, first at one place, then at a second opposite point 2 cm. above the first, and the amounts of water absorbed by them (i) after one cut, (ii) after being "doubly sawn" were estimated.

The experiments were made on branches of Portugal laurel of various sizes.

EXP. 20.

	Branch A.	Branch B.	Branch C.	Branch D.
Diameter of branch in mm.	15	12	11	14
Distance of lower, first sawn cut from the end, in cm.	13	13·5	16	13
Distance between lower and upper cuts in cm.	2	2	2	2
Depth of lower cut in mm.	9	8	6	8
Depth of upper cut in mm.	8	5·5	6	8·5
Weight of water in gramms absorbed during the first half-hour after fixing	2·38	1·62	0·97	2·3
Weight of water absorbed during second half-hour after fixing	2·28	1·32	0·86	2·04
Weight of water absorbed during half-hour interval between first and second cutting	2·23	1·23	0·83	1·93
Weight of water absorbed during half-hour after second cutting	0·35	0·45	0·26	0·45
Weight of water absorbed per half-hour during second and third half-hours after second cutting	0·43	0·60	0·32	0·59

It will be seen in all four cases that the first cut produced

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little effect, while the second cut produced a great depression followed by slight amount of recovery.

In the second series of experiments, also made on Portugal laurel, both the sawings were made at one time. It will be seen that here again a considerable diminution in the absorption of water occurred, followed by a certain amount of recovery.

The case of branch H is instructive; the two saw cuts were not made either quite parallel or quite deep enough, so that they did not completely overlap, consequently the effect of the double sawing was much less than in the other cases.

EXP. 21.

	Branch E.	Branch F.	Branch G.	Branch H.
Diameter of branch in mm.	13	12	15	17
Distance of lower from the end in cm.	19	15	13	17
Distance between lower and upper cuts in cm.	2	2	2	2
Depth of lower cut in mm.	9	8	9	8
Depth of upper cut in mm.	5	8	9	9*
Weight of water in gramms absorbed during one hour after fixing	4·31	2·85	5·39	4·96
Weight of water absorbed during 1 hour after double sawing	0·91	0·83	0·86	2·47
Weight of water absorbed per hour during next 17 hours, chiefly at night	1·12	0·68	1·12	2·16

Before discussing the experiments, a possible source of error connected with them must be pointed out. It is clear that the action of the potometer depends on the apparatus being air-

* Not parallel and therefore not across.

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tight. If a flaw were to exist in the india-rubber tubing it is evident that the air would be drawn in through it instead of water being drawn up through the thermometer tube. It may be supposed therefore that if an incision is made into the vessels of the plant the result would be the same, and no readings would be obtainable since the apparatus would no longer be air-tight. The following experiment shows that this result may actually occur. A branch of Acacia (*Robinia*) was cut under water and fitted into the potometer. The branch was cut half through, and after a minute or two a chain of bubbles was seen rushing into the potometer from the cut end of the branch, and no reading could be taken since no water was being drawn through the thermometer tube. On another occasion the same thing was observed, as shown in the following table:

EXP. 22.

p.m.	Rate
1.16	1·47
18	1·43

(i) A nick was cut down the pith:

19	0

The cut was rubbed with lard:

21	12·7

Branch sawed $\frac{1}{2}$ through opposite (i) and 2 cm. lower down; the place being covered with lard:

30	4·9

Exp. 22 also shows that a great depression follows the second incision, even when leakage of air is prevented by using lard.

Robinia was the only plant met with in which a single cut reduced the rate of absorption to zero. And it is certain that in the other plants experimented on, the great diminution observable after the second cut was not due to leakage of air into the apparatus. If this had been the case it is clear that the great fall in rate of absorption would have occurred when the first incision was made; or at any rate the falls in rate caused by the first and second cuts would have been equal. But this was not the case. The fact that recovery, i.e. an increase in the rate of absorption, usually follows the depression caused by double sawing, is a further proof that leakage is not the cause of the depression. In the case of Portugal laurel the following results show clearly that the kind of leakage which occurs in Robinia has no share in depressing the rate of absorption; and this might have been inferred from the fact that the wood consists chiefly of tracheids.

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EXP. 23. Portugal laurel cut from tree, Aug. 13, 1885.

a.m.	Rate
11.29	39·0
43	38·8

(i) Sawed half through*:

48	36·2
55	36·4
12. 8	36·2

(ii) Sawed half through at right angles to (i), leaving quarter of original transverse section intact:

p.m.	
12.12	33·8
25	36·0

(iii) Sawed so as to leave $\frac{1}{8}$th intact:

p.m.	Rate
12.29	30·7
43	33·1

(iv) Sawed so as to leave $\frac{1}{16}$th:

12.47	28·6
1. 0	30·1

(v) Made a tangential cut so as to reduce the bridge of wood through which the current passes to a piece measuring 2 mm. tangentially by 1 mm. radially:

1. 6	21·6
12	22·8

Thus the sectional area of the wood was reduced from 91 square mm. to 2 square mm., while the current was only reduced from a velocity of 39 to 22·8, i.e. from 100 to 58·5. It is clear therefore that if the leakage of air could depress the rate of absorption, that so deep a section must have produced serious depression. But it will be seen from the rest of the experiment here given that a serious depression only occurred when a small incision 2 mm. in depth was made on the opposite side to the above described deep cut.

EXP. 23 (*continued*).

Time	Rate
1.12	22·8

(vi) Made a cut 16 cm. above and opposite to (v), to the depth of 2 mm.

16	20·9
19	22·8

* At 14 cm. from the cut end of the branch. The branch was 13·5 mm. in diameter. Saw (i) was 7 mm. deep.

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(vii) Cut 2 mm. deep, 8 cm. above and opposite to (v):

22	19·8
27	21·2

(viii) Cut 2 mm. deep, 4 cm. above and opposite to (v):

30	14·3

(ix) Cut 2 mm. deep, 1 cm. above and opposite to (v):

35	7·9

Thus the large incision described in the first part of Exp. 23 only reduced the rate from 39 to 22·8, while by sawing to the depth of 2 mm. at several places on the opposite side the rate of absorption was reduced from 22·8 to 7·9.

Finally it may be pointed out that by a simple experiment it can be shown that the leakage in question does not take place even in such vascular plants as the Jerusalem artichoke (*Helianthus tuberosus*).

Exp. 24. A Portugal laurel was placed in the potometer and gave the readings:

Time	Rate
4.44	49·0
52	49·3

The india-rubber cork (*e*, fig. 1) was removed and a piece of Helianthus stem, cut square at both ends*, fixed with india-rubber tubing so as to act as a cork. If it were to act as a leaking cork the rate ought to fall. But the following readings show practically no fall:

Time	Rate
5.5	51·8
24	40·3

The india-rubber cork was then replaced with the result:

Time	Rate
5.35	40·0
43	38·0

The laurel was then removed and the piece of Helianthus fixed in the potometer, not as a cork but as the transpiring plant, to ascertain what rate of absorption could be produced by the evaporation of its free cut end:

Time	Rate
5.59	1·4

* At first 26 cm. in length then after a few minutes reduced to 13·5 cm.

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so that it is evident that when it was acting as a cork it was not adding materially to the water absorbed, and therefore did not interfere with the reading given by the Portugal laurel.

Conclusions as to Dufour's Experiments.

We may safely conclude from the result of the above experiments that the effect of double sawing on the transpiration-stream of angiosperms is great. And we can by no means agree with Dufour's statement, that the difference in absorbing power between normal and doubly-sawn branches is "at first only slight*". Dufour only gives three experiments in which the amounts absorbed by sawn and not sawn branches are compared. In two of these the difference is certainly slight (126 : 124, and 52 : 41) at first; but even in these the difference mounts up so that on the last day of the experiment the intact branch absorbed 97 cc. while the sawn one absorbed 2 cc.

§ 5. *Estimation of the effect of "double sawing" on branches through which a current of water is drawn by means of a pump, &c.*

The method employed was as follows:

A branch is fixed by its upper end to a thick india-rubber pipe (*a* fig. 2) connecting it to a water air-pump, the other end of the branch being fixed to the potometer *b*. A slight suction of the pump causes a rapid current of water through the branch, the rate being read by the potometer. The branch is then sawed at *c* and *d* and the effect on the reading of the potometer noted.

Exp. 25. Aug. 13, 1884. A branch of Portugal laurel 658 mm. in length was cut between 9 and 10 in the morning, was kept damp until it could be immersed in water in the laboratory, i.e. about half an hour after it had been cut from the tree. It was fitted to the Sprengel pump (at 11.13 A.M.) in such a way that the current of water was drawn through in a natural direction, i.e. from the basal to the apical end. The sucking power was about 10 cm. of mercury.

Time	Rate
12.0½	125·0
Sawed half through at 27 cm. from the base.	
2	83·3
33	78·1
37	Sawed half through on the opposite side at 25 cm. from the base.
42	34·7

* Sach's *Arbeiten*, p. 50.

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Exp. 26. Aug. 14, 1884. A similar experiment was made with a branch of Yew 55 cm. in length. The branch was connected to the potometer and to the Sprengel pump at 10.7 A.M. The sucking force being kept at about 6 to 6·5 cm. of mercury*.

Time	Rate
10.44	79·3
10.49–50	Sawed half through 21 cm. from the base.
11.26	78·1
41	75·2
43	Sawed opposite half at 18½ cm. from base.
47	35·7
12.8	36·4
46	37·4

In the above experiments, the cut end of the branch which was not fixed to the potometer was exposed to a rarefied atmosphere in the air-pump tube, and therefore the evaporation of this end would cause a slight *transpiration* current in the wood. It was therefore determined to immerse both ends of the branch in water; thus the negative pressure from the pump was transmitted to the wood through a small column of water.

In some cases a siphon was used instead of a Sprengel pump; this was the case in the following experiment.

Exp. 27. Aug. 19, 1885. A branch of Portugal laurel 68 cm. in length was fixed by its basal end to a potometer, at the other end was applied a siphon with a fall of 118 cm. of water.

	Time	Rate
	12.5	74·1
	15	72·5
	20	75·2
(i)	21	Sawed through $\frac{7}{13}$ of diameter, at 21 cm. from lower end
	22	46·5
	32	46·5
	25	46·9
(ii)	37	Sawed through $\frac{7}{13}$ of diameter opposite to and 2 cm. from (i)
	42	4·5

Exp. 28. Aug. 20, 1885. A Portugal laurel branch 62 cm. in length was fixed into the potometer, the upper end being attached to a siphon having a fall of 135 cm. of water.

Time	Rate
5.1	46·5
10	47·6

* We purposely employed far lower pressures than Dufour used, as they afforded readings more comparable with those observed in transpiring branches.

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(i)	Sawed through $\frac{8}{13}$ of diameter, at 24 cm. from base	
	18	39·2
	29	35·1
	40	35·1
	43	33·9
(ii)	47	Sawed through $\frac{7}{13}$ of diameter, opposite to and 2 cm. from (i)
	54	5·6
	6.18	5·0

In both these cases it is clear that the second cuts produce great depression in the rates of absorption, though the effect produced by the first cut is different in the two cases; in the second may be seen the absence of the recovery which occurs in the transpiration-stream after incisions have been made.

In the following experiments, the suction was applied by a Sprengel pump to the upper end of the branch, which was covered by a few centimeters of water.

EXP. 29. August 21, 1885. A branch of Sycamore (cut under water) about 30 cm. in length was used. The current of water was estimated by weighing the amounts of water sucked up by the branch in given intervals of time.

	Gramms.
Intact branch transmitted in half an hour (Suction = 17·5 to 18·5 cm. of mercury)	2·14
After two incisions $\frac{7}{12}$ and $\frac{8}{12}$* of diameter, it transmitted in half an hour (Suction = 18·5 to 19 cm. of mercury).....	0·63

EXP. 30. Lilac (*Syringa*). August 21, 1885. A branch 59 cm. in length, cut under water.

	Gramms.
Intact branch transmitted in $\frac{1}{2}$ hr. under a sucking force of 16 cm. of mercury	2·75
After two incisions had been made, it transmitted 1·11 gr. in 2 hrs. i.e. per $\frac{1}{2}$ hr.	0·28
The incisions (2 cm. apart) were each $\frac{7}{12}$ of diameter, and were at 23 cm. and 25 cm. from the base.	

EXP. 31. Sycamore, cut under water, Aug. 24, 1885.

	Gramms.
Intact branch transmitted per half hour under a sucking force of 17 cm. of mercury	6·59
After two incisions had been made it transmitted, per half hour	0·23

* 13·5 cm. and 15 cm. from the base.

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The incisions were sawn at 17 cm. and 19 cm. from the base on opposite sides, and to the depth of over half the diameter.

It will be seen that there is a general resemblance between the effects obtained with the pump and those obtained when the transpiring branch supplies the force, though the effect would seem to be greater in the pump-experiments. Thus in both we get a depression in rate when the first cut is made and a much greater depression following the second incision.

§ 6. *Discussion on a point of difference between the pump and transpiration-results.*

In the pump—or siphon—experiments *no recovery takes place after the depression.* This is exactly what we should expect—the force exercised by a pump is constant; whereas the force exercised by the transpiring plant can increase when anything occurs to block the passage of the stream. As soon as the current of water is diminished, the transpiration of the leaves will begin to empty the xylem elements of water, and this will necessarily increase the sucking power and will increase the current through the blocked place. This effect may be clearly seen in cases such as Sachs has described where the absorbing power of a cut branch has been lessened by lengthened immersion in water. If a fresh surface is cut and the branch quickly fitted into the potometer the current will be found to be extremely rapid. It is found that this quick rate of absorption is not permanent but is rapidly falling.

Exp. 32. A branch of Portugal laurel which had become very dirty at the cut end, and which was absorbing water slowly, gave the following readings:

July 24, 1884.	Time	Rate
	4.27	28·6
	5.11	27·8

A clear surface was cut and the branch refitted to the apparatus:

5.18	333
18½	333
23	222
32	161

The same result may be obtained if a branch is cut off above the place where two opposite incisions have been made, and this shows once more that the double sawing of a branch acts as a serious block to the transpiration-stream.

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Exp. 33. Aug. 25, 1885. Portugal laurel.

Time	Rate
1.5	61
2.35	65
2.40	Sawn more than half through on opposite sides, at 2 cm. apart
2.50	16
3.11	19
5.16	27

Branches severed above the double saw-incisions, and remounted:

5.20	137
24	143
30	93
40	67
47	69

Exp. 34. Another experiment on Portugal laurel may be quoted, which confirms the last.

Aug. 25, 1885. Time	Rate
2.42	83
45	Branch sawn more than half through at 2 cm. apart on opposite sides
52	19
3.9	23
5.29	32
35	Cut off 14·5 cm. above the upper incision, and remounted
38	182
48	116
57	96
6.8	78

Lastly the same thing was proved by weighing the amounts of water absorbed in given times.

Exp. 35. Two branches of Portugal laurel which had been doubly sawn on the previous day, and had remained in this condition all night, were severed above the double saw-cuts, and were again allowed to absorb water by this new surface.

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	Plant i. Gramms.	Plant ii. Gramms.
Weight of water absorbed during one hour before the branches were doubly sawn	4·31	2·85
Weight of water absorbed during one hour after sawing	0·91	0·83
Weight of water absorbed during one hour after the branch had been cut off above the saw cuts	6·86	4·91

It will be seen that the difference pointed out between the transpiration- and filtration-streams is quite in accordance with the intracellular theory of water transport, though it may also perhaps be consistent with what we know of imbibition.

§ 7. *Experiments with solution of eosin and with water containing coloured particles in suspension.*

The adherents of the inbibition theory naturally look with suspicion on experiments made with coloured fluids, but to others they are not without interest.

If a transpiring branch be placed in a solution of eosin, the colour as is well known gradually spreads over the whole specimen, so that the leaves become discoloured and the wood of the smallest twigs shows a bright pink colour. But if the branch has been doubly sawn before it is placed in the eosin a very characteristic appearance is produced. Fig 3 is a diagram representing the appearance of such a branch. Above the "double saws" the colour shows in two narrow strips running close behind the uppermost saw-cuts, that on the side nearer to the observer being shown at *a*. Below the two saw-cuts is a strip of tissue *b* corresponding to *a*; this spreads out until lower down at a point not shown in the diagram the whole branch has become coloured. *Between* the saw-cuts is a belt of colour marked by a horizontal arrow which unites the streams *a* and *b*.

The more usual appearance is shown in fig. 4 where the colour is fairly spread over the whole surface of the branch below the lower cut, but where above the upper saw-cut the two strips *a*, *a*, are seen. In a transverse section these two streams may be seen as two wedge-shaped areas of colour.

The course of the stream may also be shown in another way. If the cut end of a transpiring branch be immersed in water containing coloured particles in suspension, the cut surface rapidly assumes a bright tint of the colour employed.

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If the branch has been previously "double-sawn" it will not (at least at first) be uniformly coloured, but will show two somewhat wedge-shaped patches of colour each of which corresponding in position to one of the lines of colour in figs. 3 and 4. These are shown in fig. 5. It will be seen that one side of the wedge vertically below the inner edge of the lower saw-cut is sharply defined, while the other edge is feathered. This shows that the suction is strongest where the stream passes by the edge of the saw-cut, and gets weaker along radii further removed from this point.

The chief point of interest connected with this subject is that similar appearances can be produced by forcing eosin solution through doubly sawn-branches; it is impossible to distinguish such a specimen from a transpiring branch which has been doubly sawn and placed in eosin. The wedge-shaped patches of colour, with one feathery edge, as seen in fig. 5, may also be produced by attaching a double sawn branch to a pump while the other end is in a mixture of carmine and water.

§ 8. *Conclusion on Dufour's Experiments.*

We have not enough data to criticise Dufour's paper. Elfving* and Scheit† have pointed out a possible source of error from the effect of air on the cut ends of the branch. We have shown the importance of great care in making the saw-cuts; it is easy (even without carelessness) to leave a small region of transverse section unsevered, and we have shown how readily a current may be conveyed over a very narrow bridge of wood. Both these sources of error may have contributed, in different ways, to leading Dufour to an untenable result. That it is untenable the following recapitulation of our results seem to prove.

I. Double sawing does produce a great depression in the rate of absorption. No one who will make a single careful potometer experiment (with an Angiosperm) will be able to doubt it for a moment.

II. Double sawing does *not* absolutely interrupt the filtration current‡.

[Absolute stoppage of the current is required for Dufour's argument, since the withering experiments which are coordinated with the pressure experiments in the argument give no quantitative record of the amounts of water passing by the saw-cuts.]

III. Double sawing produces a depression in the filtration current similar to, though possibly greater than, the depression caused in the transpiration current.

* *Bot. Zeitung*, 1884, No. 13.

† *Acta Soc. Fennicæ*, t. xiv. 1884.

‡ Scheit has shown this fact, loc. cit.

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IV. The experiments with coloured fluids tend to confirm our belief in the similarity between the transpiration and filtration streams.

We have hitherto considered our results chiefly with reference to Dufour's experiments. We will now discuss them in a more general manner.

An imbibitionist might argue that since the normal direction of the stream in wood is longitudinal, it naturally cannot be transmitted by imbibition so easily in a transverse direction. We have too little experimental knowledge of the nature of imbibition to enable us to criticise this argument on general grounds. But our experiments do yield an argument against the view in question, though not one of great value.

We have shown that *on the average* the depression in the rate of absorption is considerably greater in Angiosperms than in Gymnosperms. This at least seems to be true for Taxus as compared with Angiosperms: the results obtained with Pinus being possibly too discordant to be trusted. If on general principles it is to be assumed that water of imbibition cannot travel easily in the transverse direction we must assume it to be true for all kinds of wood. There is no reason why it should be applicable to Angiosperms and not to Gymnosperms. Therefore "double sawing" ought according to the imbibitionists to produce the same effect on Gymnosperms as on other trees—and this is not the case. But it must be added that we are not able quite clearly to explain the difference in question by any theory of water-transport.

It is clear, since double sawing destroys the vessels as carriers of water, that therefore the operation in vascular plants is equivalent to destruction of a large part of the water-channels. And these channels are of large bore and without transverse divisions, in other words the destroyed channels are those best fitted for longitudinal transmission. But in wood made up of tracheids as is that of Gymnosperms, this destruction of part of the water-course does not take place.

But it may be said that our experiments in which a *single* deep cut was made, show that the destruction of a very great part of the water-channel does not seriously depress the rate. Hence it may be argued that the chief cause of the depression in the double sawing experiments, is rather to be sought in the fact that the current has to travel transversely than in the fact that it has a diminished amount of tissue to travel in.

But the above-mentioned remarkable fact (Exp. 23) that a deep single cut does not seriously depress the rate shows that transverse transmission is easily effected if there is given a suf-

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ficient length of branch in which transverse diffusion of the current may be effected. In other words the current can flow across the branch if the obliquity of flow imposed on it is slight.

The following experiments bear on this point:

EXP. 36. Portugal laurel. Aug. 14, 1885.

The first incision I. was at 11 cm. from the cut end. Incisions Nos. II., III., IV., V., VI. were then made successively at distances of 10, 8, 6, 4, 2 cm. from I., as shown in the diagram, Fig. 6. The incisions were 7 mm. deep, the branch being 13—14 mm. in diameter.

Time	Rate
11.50	26
54	27
57	27
Sawed at I,	
12.1	24
6	25
15	25
Sawed at II,	
21	13
30	17
35	17
Sawed at III,	
39	17
55	18
Sawed at IV,	
58	17
1.36	18
4.8	19
Sawed at V,	
4.13	16
35	17
Sawed at VI,	
4.45	about 5·5

We see that when the branch was sawn on the side opposite to the first cut, even though at 10 cm. distance, a considerable depression occurred. That is to say, a considerable depression was caused by the current being forced to travel obliquely. The depression was not markedly increased until the obliquity of transmission was greatly increased by sawing at 2 cm. from the first cut.

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Exp. 37. The following experiments, also on Portugal laurel, confirms the above result. The first cut (I.) was at 10 cm. from the base, II., III., IV., V. were on the opposite side at distances of 10, 7, 4, 2 cm. from I.

Sept. 1, 1885

Time	Rate
a.m.	
11.59	33
12.6	34
Sawed (I),	
18	32
22	33
12.25 Sawed (II),	
28	15
51	20
54	Sawed (III)
59	13
1.30	16
32	Sawed (IV)
37	11
46	12
49	Sawed (V)
56	5

In the following table the amount of diminution in the rate of absorption produced by the cuts is expressed as a percentage of the rate recorded before each cut was made.

Number of Incision (See Diagram)	I	II	III	IV	V	VI
Depression produced (1st Experiment)	11	48	0	5	16	68
Depression produced (2nd Experiment)	6	54	35	31	58	

The point in which the experiments confirm each other is that the big effects are produced by the second cut, and by the last cut. That is to say (i) when the current is thrown across the line of the branch, and (ii) when the obliquity reaches the amount produced by a distance of two centimeters between the cuts.

Elfving* has explained the difficulty experienced in forcing water under pressure through coniferous wood in a tangential direction. The same explanation is applicable to the double sawing experiments. Fig. 7 represents a "doubly sawn" branch.

* *Bot. Zeitung*, 1882.

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FIG. 3.

FIG. 4.

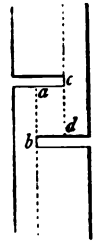

FIG. 7.

FIG. 5.

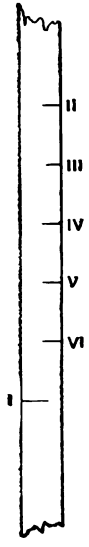

FIG. 6.

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The current is supposed to flow upwards, and in passing from the line *ab* to *cd* has to traverse a large number of partitions, formed by the longitudinal walls of the elements in which the current flows. The longer *ac* is, i.e. the more the saw-cuts overlap, the greater is the number of such walls and the greater the block in the current. Again, the shorter *ab* is, the fewer will be the pits through which the current can find its way; and this may account for the facts given in Experiment No. 35, where the amount of diminution of the current is shown to be much greater when *ab* is 2 cm. than when it is 10 cm.

It may be urged that all these arguments apply to wood such as that of the Yew, consisting entirely of tracheids, as well as to angiosperm wood, and that therefore we ought to have the same effect in gymnosperms as in angiosperms.

The tracheids of the Yew are at least 70 or 80 times as long as they are broad, so that in travelling transversely the length of a single tracheid the water current has to traverse 70 cell walls instead of one. So that it is difficult to see why double sawing produces so slight an effect in the Yew. It must however be remembered that in the case of wood consisting altogether of tracheids, a transverse current flows through the same elements as a longitudinal current. It still travels from tracheid to tracheid. But in angiosperm wood the current when forced to travel transversely no longer travels in its former elements, since the vessels are destroyed as channels, but must travel chiefly in the thick-walled (what is more important) scantily pitted wood-cells. So that it is unable to travel laterally with any ease unless it has a long space to move transversely in.

On the whole we think that the difference which probably exists between angiosperms and the Yew or Scotch Fir is to be explained by the fact that after the operation of double sawing the only tissues left are badly fitted for transmission, while in the case of the Yew or Fir no tissue is rendered incapable of transmission. Further experiments would be needed to clear up the point satisfactorily.

§ 9. *Experiments on bending and compressing the tissues of transpiring branches.*

Dufour showed that twigs bent sharply on themselves, so that the apical and basal halves are parallel, do not wither, nevertheless he found that such branches are highly impervious to a stream of water under pressure. Hence, as before, he argues that the transpiration-stream does not flow in the cavities of the elements, for if it did so, it would be hindered in the same way as the filtration-stream is blocked by the process of bending. We proceed to show

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that the transpiration-stream can be hindered by treatment such as bending or compressing the branch.

EXP. 38. Sept. 18, 1884. Bramble (*Rubus sp.*), cut under water.

Time	Rate
1.0	40·8
bent to an angle of 20°	
3	34·8
unbent	
4	58·9
4.6	37·0
bent nearly parallel	
13	29·4
unbent	
15	50·2
26	37·7

These experiments show that (i) bending does depress the rate of absorption; (ii) when the stem is straightened again, the rate is temporarily quickened by the removal of the blocking caused by the bend.

EXP. 38. Continued Sept. 18, 1884.

Time	Rate
4.44	34·7
Stem bent twice and tied closely in this position	
52	18·5
unbent	
54	57·1 (?)
56	44·6
5.25	32·8

The bending and straightening were again repeated with precisely similar results, viz.:—

Rate	
32·8	straight
18·5	bent
56·8	straight.

These results confirm what was stated above. They are plainly explicable on the theory that the transpiration-stream travels in the *cavities* of the wood.

They also confirm Russow's statements that bending does not absolutely stop up the lumen of the vessels.

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Since bending (even twice) obviously did not seriously diminish the lumen of the vessels we tried squeezing the branch with a vice*—half the branch being cut away in order that the vice might be used more conveniently.

EXP. 39. Sept. 17, 1884. Bramble.

Time	Rate
5.52	29·1
Vice screwed up tight	
57	11·1
Vice slackened	
59	35·9
Vice screwed up	
6.6	16·7
Vice slackened	
6.7	43·9

EXP. 40. Sept. 20, 1884. *Helianthus tuberosus.* Conditions of experiment similar in all respects to those of the last.

Time	Rate
5.28	45·7
Vice tightened	
32	12·0
Vice loosened	
33	74·6
38	53·4

Here again we get a considerable depression; followed by a temporary increase when the block is removed.

It may be argued by an imbibitionist that the severe pressure exercised by the vice may have injured the cell-walls by crushing them and thus interfered with their transmitting qualities.

The fact that the rate recovers when the vice is loosened shows that the path of transmission, whatever it may be—cell-wall or cavity—is not permanently injured, and it is easier to believe that the lumen of a tube might recover its previous size, when the pressure is removed, than that a crushed cell-wall should recover its original properties. It may be added that the doctrine of imbibition seems to exclude the possibility of pressure affecting the

* Vesque found that a willow branch could be made to wither by squeezing it in a vice (*Ann. Sc. Nat.* 1884). Kohl has published experiments similar to ours. *Bot. Zeitung*, 1885.

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water of imbibition. Thus Dufour says: "Als Grundsatz gilt: das in Zellwändern imbibirte Wasser ist durch gewöhnliche Druckkräfte überhaupt nicht verschiebbar."

It seems to us that the above results of experiments on bending and squeezing are only explicable on the theory that the chief stream of transpiration is in the cavities and not in the cell-walls.

We cannot understand the great difficulty found by Dufour in forcing water under pressure through bent twigs. In the only experiments which we made on this point the difficulty was not found to be so great.

A bramble 213 c.m. in length, was cut under water, and was fitted to the potometer by its basal end, while the apical end was attached to the water air-pump. It was subjected to a suck of 60 to 65 c.m. of mercury and the readings of the potometer taken at various pressures. Then the branch was bent in two places, the bends being tied as closely as possible. The result was that with a pressure of 60 c.m. mercury the potometer rate for the unbent branch was 45·4; for the bent branch 19·0; or as

Straight	100
Bent	41·9

In another similar experiment the result was greater. The branch was 76 c.m. in length: the pressure varied between 29·5 and 30·5 c.m. of mercury.

	Rate		
Before bending	60·9	or	100
Doubly bent	13·6		22·3
Straightened	61·3		100·7

These experiments show the similarity that exists between the transpiration and filtration currents, and they can only be explained on the supposition that the water of transpiration travels in the cavities of the wood-elements.

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PROCEEDINGS

OF THE

Cambridge Philosophical Society.

February 1, 1886.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

M. M. Pattison Muir, M.A., Caius College.
A. B. Basset, M.A., Trinity College.
A. H. Evans, M.A., Clare College.
T. Brill, M.A., St John's College.
F. G. Heathcote, M.A., Trinity College.
Dr C. Olearski was elected an associate.

The following communications were made to the Society:

(1) *On a fall of temperature resulting from an increase in the supply of heat.* By G. D. LIVEING, M.A.

In the course of some experiments on the formation of ethyl iso-cyanide by the action of potassium cyanide on oxalic ether my assistant, Mr Robinson, observed that when the gas flame employed to heat the oil-bath in which the mixture was digesting was turned out the temperature of the mixture rose while that of the bath fell. On again heating up the bath the temperature of the mixture fell to rise again when the bath cooled.

The apparatus used was simple. The mixture was placed in a flask to the mouth of which was fitted, by means of a cork, a reversed condenser, so that any distillate could flow back into the flask. Through the cork two thermometers were passed, one dipping into the mixture in the flask, the other giving the temperature of the vapour above the mixture. The flask was placed in a bath of oil, or in later experiments of paraffin, and was held in such a position that the level of the oil, or paraffin, outside was the same as that of the mixture inside the flask. A third thermometer was

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dipped into the oil or paraffin and gave the temperature of the bath. The bath was heated by a Bunsen burner beneath it.

The oxalic ether and the potassium cyanide were mixed in proportions intended to correspond to two molecules of the latter to one of the former, so as to react according to the following identity,

$$(C^2H^5)^2C^2O^4 + 2KCN = 2C^3H^5N + K^2C^2O^4.$$

Commercial cyanide was employed, which was afterwards found to contain a great deal of carbonate. The effect of this was equivalent to having an excess of oxalic ether.

As the bath was heated all the three thermometers rose gradually until that in the paraffin indicated 150°, when the thermometer in the mixture stood at about 108° and that in the vapour at about the same degree. The mixture at this temperature boiled rapidly, and the thermometer immersed in it shewed rapid fluctuations between 90° and 110°, which were no doubt due to convection currents, and to the return of the condensed distillate to the flask. We seemed to have a substance evaporating which had a boiling point not far from 100°. The boiling point of ethyl-cyanide or propio-nitrile, one of the two isomers which have the composition C^3H^5N, has been given by different observers at various degrees between 82° and 98°·5, and it seems probable that the higher figure is the more correct. Gautier, using a carefully purified sample, found its boiling point to be 96°·7 (*Bull. de la Soc. Chem.* t. IX. p. 4). Oxalic ether does not boil below 180°. It is therefore most likely that the vapours evolved at this stage consisted principally of propio-nitrile.

When the heating was continued and the paraffin reached 168° the temperature both of the mixture and of the vapour above it fell suddenly to 85°, and that of the vapour soon dropped to about 74°. As long as the temperature of the bath was kept at about 170° the mixture remained at a temperature between 85° and 90° and the vapour at about 74°.

When the lamp under the bath was now turned out, the paraffin of course gradually cooled, and when it had reached 150° the mixture had risen to 101° and the vapour to 90°. On again heating up the bath the fall in temperature of the mixture and vapour was observed to occur as before when the paraffin reached 168°; and the same phenomena recurred several times when the bath was alternately cooled and heated.

It seems, at first sight, strange that the rise of temperature of the bath, and consequent increase in the supply of heat, should cause a fall in the temperature of the mixture. But I have no doubt that it is due to a difference in the chemical changes which occur in the mixture at the different temperatures. When the bath is at about 170° we seem to have a substance formed which

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has a boiling point somewhere between 74° and 85°. This substance I take to be chiefly ethyl iso-cyanide which boils at 78°. At any rate the distillate smells strongly of that substance. The mixture does not give the liquid with the lower boiling point, at least in any quantity, until the bath reaches 168°. We may suppose that the flask and the layer of the mixture next the bottom and sides reach about the same temperature as the bath, and that this layer is then resolved into the iso-cyanide and potassium oxalate. This may possibly be accompanied by a storage of some of the energy supplied, but whether that be so or not, the evaporation of the volatile compound would depress the temperature of the bulk of the liquid. When the bath falls below 168° only the less volatile liquid is formed in quantity, and the mixture rises to the boiling point of the latter. In both cases the temperature of the mixture is kept down by the evaporation of a volatile compound formed by the action of heat upon it, but at the higher temperature the compound formed has the lower boiling point.

Although I have no doubt that the compounds formed are propio-nitrile and the iso-cyanide, this has not been proved yet. The distillates are by no means pure single substances, or easy of purification, so that the interesting chemical question as to the formation of propio-nitrile at one temperature and of its isomer at a higher is not yet fully solved. We know from Thomsen's researches that the heat of formation of propio-nitrile is negative, that is to say it is formed with a storage of energy; but I have not been able to meet with any determination of the heat of formation of the iso-cyanide. Considering the large storage of energy in the formation of hydro-cyanic acid it is probable that the storage in the iso-cyanide exceeds that in propio-nitrile.

It occurred to me that, if the explanation I have given of the changes of temperature were correct, similar phenomena must present themselves in other cases where chemical reactions producing compounds more volatile than the reagents are determined by particular high temperatures. For example common ether is produced by the reaction of alcohol on ethyl-sulphuric acid at a temperature which is given at about 145°. At temperatures below that the distillate is chiefly alcohol. This case is not so simple as the former because when ether is formed water is also formed at the same time. Nevertheless until the bath gets hot enough to determine the formation of ether the temperature of the mixture and its vapour will be dependent on the boiling point of alcohol, while when the bath gets hot enough ether and water will be formed, and the lower boiling point of ether will determine a lower temperature of the parts of the mixture not near the sides of the flask and as well as of the vapour. To test this I asked Mr Robinson to make an experiment with alcohol and sulphuric acid in the

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same apparatus which had been used for the cyanide. The result corresponded with anticipation. The thermometers in the bath and in the flask all rose together until the bath was at 147°, the mixture at 110° and the vapour at 75°. When the bath reached 154° the mixture fluctuated from 108° to 115° and the vapour was at 77°, but on raising the temperature of the bath to 155° the thermometer in the mixture fell suddenly to 90° and that in the vapour to 65°. It has long been recognised that in the continuous process, whereby a small quantity of sulphuric acid gradually converts a large quantity of alcohol into ether and water, the continuity of the reaction must be due to differences of temperature in different parts of the liquid. The directions for making ether given in handbooks of chemistry direct that the mixture is to be maintained at about 140°. This will certainly imply a much higher temperature for the retort and for the layer of liquid in contact with it. Our experiment seems to indicate that the reaction between alcohol and ethyl-sulphuric acid by which ether is produced begins at 155°, or at least begins to occur quickly at that temperature. With the bath at 154° the temperature of the vapour was nearly that of boiling alcohol, and it dropped when the ether began to come; but of course it would never drop to the boiling point of ether because as much water as ether is formed in the reaction and both are vapourised together. Moreover Thomsen's observations shew that this reaction is attended with a sensible evolution of heat.

(2) *On the functions inverse to the second elliptic integral.* By J. W. L. GLAISHER, M.A.

Consider the function inverse to ez x, where ez x is the Jacobian form of the second elliptic integral given by the equation

$$\text{ez } x = \int_0^x \text{dn}^2 x\, dx.$$

Let this function be denoted by ea x, so that, if

$$\int_0^x \text{dn}^2 x\, dx = u, \text{ then } x = \text{ez}^{-1} u = \text{ea } u,$$

the letter "a" in the functional sign ea suggesting the word amplitude.

It follows that, if

$$\int_0^\phi \Delta(\phi)\, d\phi = u, \text{ then } \phi = \text{am ea } u,$$

and if

$$\int_0^x \frac{\sqrt{(1-k^2x^2)}}{\sqrt{(1-x^2)}}\, dx = u, \text{ then } \phi = \sin \text{am ea } u.$$

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It is convenient to denote am ea u by ame u and to use sne u to denote sin ame u = sn ea u = sin am ea u, cne to denote cos ame u, &c. Denoting $E' - K'$ by I', the function ea x is such that

$$\text{ea}\,(x + 2E) = \text{ea}\,x + 2K,$$
$$\text{ea}\,(x + 2iI') = \text{ea}\,x - 2iK',$$
$$\text{ea}\,(x + 2E + 2iI') = \text{ea}\,x + 2K - 2iK'.$$

Taking the am of these equations we find

$$\text{ame}\,(x + 2E) = \text{ame}\,x + \pi,$$
$$\text{ame}\,(x + 2iI') = -\,\text{ame}\,x + \pi,$$
$$\text{ame}\,(x + 2E + 2iI') = -\,\text{ame}\,x.$$

The function ame x is therefore periodic with respect to $4E + 4iI'$ and quasi-periodic with respect to $4E$ or $4iI'$.

Taking the sn, cn, dn of these equations we find that sne x, cne x, dne x are doubly periodic, the periods being $4E$ and $4iI'$.

Corresponding to the formulae in Elliptic Functions in which the argument is increased by a quarter-period we have the equations:

$$\text{ea}\left(x + E - k^2\frac{\text{sne}\,x\,\text{cne}\,x}{\text{dne}\,x}\right) = \text{ea}\,x + K,$$
$$\text{ea}\left(x + iI' - \frac{\text{sne}\,x\,\text{dne}\,x}{\text{cne}\,x}\right) = \text{ea}\,x - iK',$$
$$\text{ea}\left(x + E + iI' + \frac{\text{cne}\,x\,\text{dne}\,x}{\text{sne}\,x}\right) = \text{ea}\,x + K - iK',$$

so that, for example, corresponding to

$$\text{sn}\,(x + K) = \frac{\text{cn}\,x}{\text{dn}\,x},$$

we have

$$\text{sne}\left(x + E - k^2\frac{\text{sne}\,x\,\text{cne}\,x}{\text{dne}\,x}\right) = \frac{\text{cne}\,x}{\text{dne}\,x}.$$

In general, in results in Elliptic Functions in which the arguments are u, v, $u+v$ we may replace sn, cn, dn's by sne, cne, dne's if we replace the argument $u+v$ by $u+v-E_{u,v}$ (leaving the arguments u, v unaltered), where $E_{u,v}$ is a certain function of the sne, cne, dne's of u and v.

The author had considered not only the functions ea x, ame x, sne x, &c. but also the complete system of functions ia x, ga x, ea x, ami x, amg x, ame x, sni x, sng x, sne x, &c. obtained by inverting

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the integrals

$$\mathrm{iz}\, x = \int_0^x (-k^2 \mathrm{sn}^2 x)\, dx,$$

$$\mathrm{gz}\, x = \int_0^x k^2 \mathrm{cn}^2 x\, dx,$$

$$\mathrm{ez}\, x = \int_0^x \mathrm{dn}^2 x\, dx;$$

of these the functions depending upon $\mathrm{gz}\, x$ are more regular and symmetrical than those depending upon $\mathrm{iz}\, x$ and $\mathrm{ez}\, x$. The two periods of the doubly-periodic functions $\mathrm{sng}\, x$, $\mathrm{cng}\, x$, $\mathrm{dng}\, x$ are $4G$ and $4iG'$ (where $G = E - k'^2K$, $G' = E' - k^2K'$), corresponding exactly to $4K$ and $4iK'$ in the case of the elliptic functions.

(3) *On the movement of solids through ether.* By Mr A. H. Leahy.

(4) *On Mr Galton's anthropometric apparatus at present in use in the Philosophical Library.* By H. Darwin, M.A., and R. Threlfall, B.A.

The authors exhibited and described some of the apparatus used by Mr Galton for his anthropometric measurements, and then gave an account of improvements and modifications which they had suggested in some of the instruments.

March 1, 1886.

Prof. Foster, President, in the Chair.

The President explained that the meeting on February 15 had not been held in consequence of the funeral of Mr Bradshaw, and referred to the loss the University had sustained by his death.

The following communications were made to the Society:

(1) *On the Development of the Nervous System in Petromyzon fluviatilis.* By A. E. Shipley.

The author commenced his communication by describing the formation of the neural cord from a solid keel of epiblast. He pointed out that the central canal arose by the separation of the cells in the middle line, and that the canal is not lined by cells involuted from the epidermic epiblast, as previous observers have stated. The formation of the primary vesicles of the brain was then described; and the origin of the 2nd, 5th, 7th, 9th and 10th

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nerves was considered. The ganglia on the 5th and 7th nerve fuse with the skin, and the latter nerve branches round a gill slit which subsequently disappears. The next gill slit is the most anterior in the adult. Hence the most anterior gill slit of the Lampreys corresponds with the first branchial gill of higher forms. The Spiracle is thus not represented in Petromyzon.

(2) *On the changes undergone by the proteid substances of seeds during germination.* By J. R. GREEN.

The writer, after alluding to the work of Vines and others, whereby the nature of the proteid reserve materials in seeds had been satisfactorily ascertained, and to the undoubted presence of proteid matter in the growing parts of the young plant, derived from such reserve store, touched upon the theory held by many botanists that the changes wrought in the reserve proteids to enable them to be transported through the plant are brought about by ferment action. He described investigations made by himself upon the germinating seeds of *Lupinus hirsutus* which resulted in demonstrating there the presence of such a ferment. This body was able to break up fibrin in the same way as the digestive juices do in the animal body, though its action was much less energetic.

March 15, 1886.

PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. J. H. Teall, M.A., St John's College.
S. F. Harmer, B.A., King's College.

The following communications were made to the Society:

(1) *On the Cerebral Circulation.* By Prof. ROY, M.A., and C. S. SHERRINGTON, B.A.

(2) *On a new method of Detecting Bromides; a case of so-called Catalytic Action.* By H. J. H. FENTON, M.A.

In former papers* it has been shewn that sodium hypochlorite is without action on sodium carbamate in presence of sodium hydroxide, whereas sodium hypobromite at once decomposes it with evolution of nitrogen.

$$2CONH_2ONa + 3NaBrO + 2NaOH = 2CO(ONa)_2 + 3NaBr + 3OH_2 + N_2.$$

* *Chem. Soc. Journ.* xxxv. 12. Royal Society, Dec. 1885.

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Carbamates differ in this respect from all other substances yet studied, and the actions serve to detect and estimate them in presence of ammonia, urea, &c.

It appeared probable therefore, that if a soluble bromide were added to a mixture of sodium carbamate and hypochlorite, nitrogen should be evolved, by reason of the hypobromite formed in solution* potentially or actually. This is found to be the case.

A solution of sodium carbamate may be prepared

1st. By dissolving ammonium carbamate in a strong solution of sodium hydroxide, and allowing the mixture to stand for one or two days over strong sulphuric acid under a bell-jar.

$$CONH_2ONH_4 + NaOH = CONH_2ONa + OH_2 + NH_3.$$

2nd. By dissolving ammonium carbamate in solution of sodium hydroxide and adding a slight excess of sodium hypochlorite.

$$2CONH_2ONH_4 + 3NaClO + 2NaOH = 2CONH_2ONa + 3NaCl + 5OH_2 + N_2.$$

3rd. By Drechsel's method, namely by adding an alcoholic solution of sodium ethylate to ammonium carbamate dissolved in aqueous ammonia.

If to a solution of sodium carbamate, prepared by any of these methods, an excess of sodium hypochlorite (and, if necessary, sodium hydroxide) be added, and the liquid shaken until bubbles of gas (due to traces of ammonia) cease to be evolved, the mixture may be kept for hours with hardly appreciable decomposition; but the addition of a soluble bromide will cause a copious evolution of nitrogen—immediately if the quantity of bromide is considerable, and more slowly with small quantities.

This reaction, then, serves to detect bromides in presence of unlimited quantities of chlorides, hypochlorites &c. Iodides appear to give a slight action, but so slowly in comparison with bromides that it seems not improbably due to contamination with bromide. I have not succeeded, however, in obtaining a specimen of iodide which gives no action at all.

Where *traces* of bromide have to be looked for, it is advisable to make a blank comparative test, since there is usually a very slight decomposition of the carbamate solution, which might be misleading. Equal volumes of a solution of sodium carbamate, sodium hypochlorite in excess, and sodium hydroxide, are placed in two long tubes. The suspected liquid added to one, and an equal volume of distilled water to the other. The tubes are shaken and allowed to stand for some time (10 minutes to 1 hour according to quantity). On again shaking, a few bubbles only will appear in the tube containing the blank test, whilst if bromide is present there will be a considerable effervescence in the other tube, due to evolution of nitrogen.

* Lyons, *Pharmaceutical Record*, May, 1885.

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In this way it is quite easy to detect one part of bromine in 6000 parts of water. Probably a much smaller quantity would be apparent.

If the decomposition is due to the formation of hypobromite, it is evident that bromide will be again produced by its action on the carbamate, and provided that there is always excess of hypochlorite present the action should be continuous.

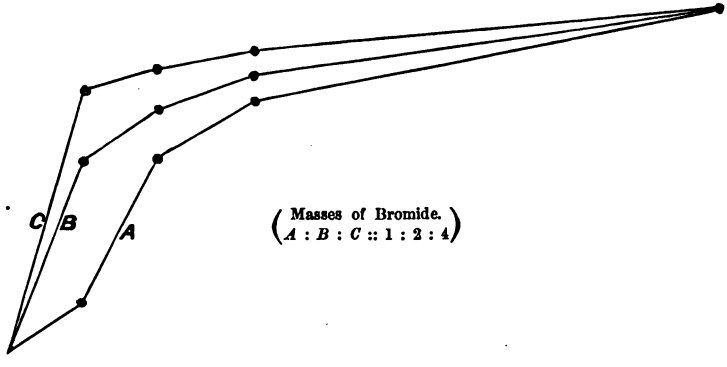
The total amount of decomposition does, in fact, appear to be independent of the mass of bromide present, at any rate within very wide limits—i.e. any appreciable quantity of bromide would be able to cause the decomposition of an almost unlimited amount of carbamate in presence of hypochlorite.

Thus ·3854 grams of potassium bromide readily caused the evolution of 340 c.c. of nitrogen from the above mixture, and the resulting solution was still strongly "active"—i.e. at once caused an evolution of nitrogen from fresh portions of the mixture. This quantity of bromide in the form of hypobromite alone, could only have caused the evolution of 24·07 c.c.

The initial *rate* of decomposition however is evidently a function of the mass of bromide present (at any rate within certain limits). A rough preliminary experiment, in this direction, was made as follows:

A mixture was prepared of sodium carbamate with excess of sodium hypochlorite and hydroxide. Three equal volumes of this solution were placed in separate vessels provided with stoppers and delivery tubes*.

5 c.c. (A), 10 c.c. (B), and 20 c.c. (C) of an arbitrary solution of potassium bromide were introduced into small tubes, the first and



second being diluted with 15 c.c. and 10 c.c. of water respectively so as to make the total volumes equal. These small tubes were then

* *Chem. Soc. Journ.*, July, 1878.

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placed in the larger vessels containing the carbamate &c., and the operation conducted in the usual manner.

After 3 minutes	A	gave 10	c.c. Nitrogen
	B	— 39	—
	C	— 55	—
After 6 minutes	A	— 40	—
	B	— 51	—
	C	— 59	—
After 10 minutes	A	— 51	—
	B	— 57	—
	C	— 62	—
After about 30 minutes	A	— 67	—
	B	— 68·5	—
	C	— 68	—

The action is therefore much more rapid in the first instance with the greater mass of bromide, but as the action proceeds the weaker solution gains on it, and eventually all yield the same final result.

It appears therefore that this action of bromides belongs to the type of certain so-called Catalytic actions in which the 'activity' of the agent is explained by an intermediate stage of little stability. We have in this instance perhaps stronger evidence of such a stage than is often the case in many actions which are similarly explained.

(3) *On a Self-recording Barometer.* By HORACE DARWIN, M.A.

In all self-recording apparatus there must be some means always at hand ready to perform the work required to keep the record. In the case of a barometer a clock is used in conjunction with some other motive power; this may be the electric current from a battery; the radiation from a lamp acting on photographic paper; or the work obtained from the change of the atmospheric pressure itself. The chief interest in the instrument shown in the figure is the unusual source of energy employed; the pressure of the ordinary gas supply being made to do the necessary work. This is undoubtedly an extravagant motor, but in cases like the one before us the work required to move the mechanism is extremely small, and the cost of the gas used is quite insignificant. The ease with which the energy can be applied and the handiness of the source makes this method most useful for a great number of cases where governing and automatic regulating mechanism is required.

The gas from the main passes into the pipe *A* through a stop-

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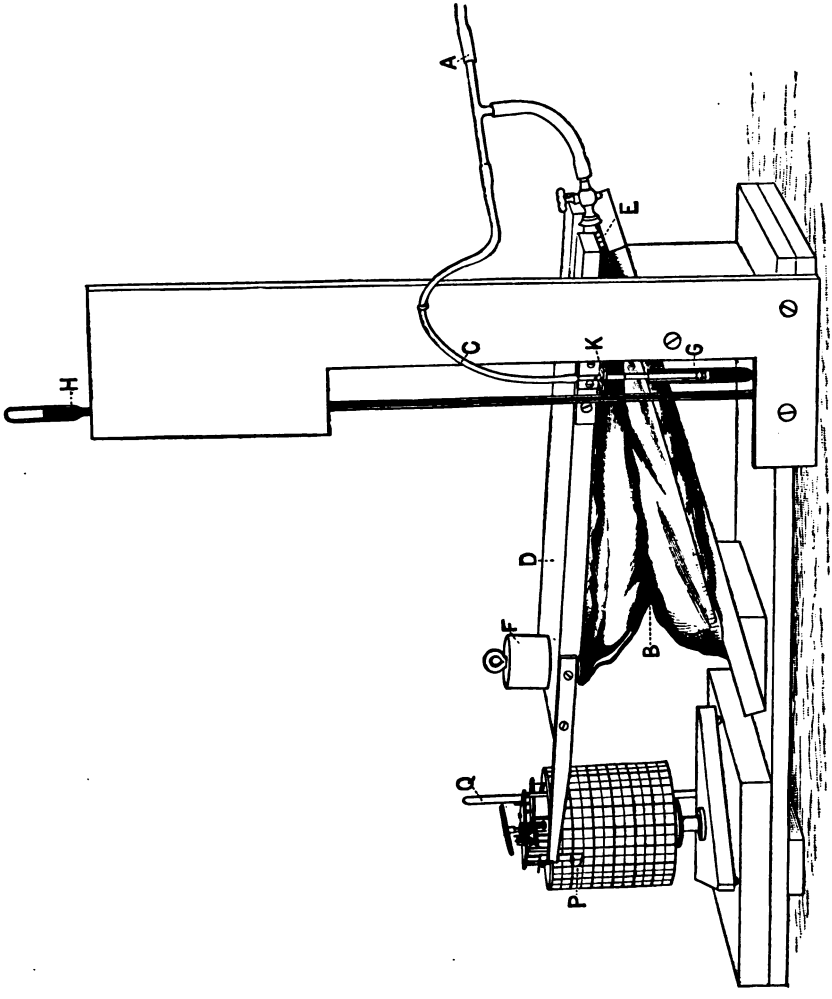
cock not shown in the figure. The stop-cock is regulated so as to allow a very small quantity of gas to pass. The pipe branches,



and the gas can either pass into the indiarubber bag *B* or else along the tube *C*. If this tube is closed, the gas will enter and expand the bag and compel the board *D* to turn about the hinges *E*. The pressure of the gas is more than sufficient to counterbalance the weight at *F*; consequently a certain amount of work can be done during the upward movement; this gives the requisite power to move the pen *P*. If more gas could pass along the tube

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C than had entered by *A*, the gas bag would empty and the pen would fall. Thus by opening and closing the tube *C*, the board can be made to rise and fall, and to move the pen with it. The barometer performs this function in such a manner, that the vertical movement of the pen is three inches upwards for every fall of atmospheric pressure corresponding to a head of one inch of mercury. The barometer is in the form of an inverted siphon, *G*, *H*. The tube *C* is connected with a small glass tube passing down the open limb of the barometer at *G*; it is supported by a bracket *K* fixed to the pressure board *D* at one-sixth the distance of the pen from the hinges *E*. Now suppose the end of the glass tube almost touches the surface of the mercury in the open limb of the barometer, and that the flow of gas through this small opening equals the flow entering by the tube *A* from the main; then no gas will enter or leave the bag and the pressure board *D* will remain at rest. When the mercury in the open limb of the barometer rises the leak will be diminished: gas will enter the bag and will raise the pressure board, together with the pen and the glass tube passing down the open end of the barometer; this will at once increase the leak and will shortly again establish equilibrium. The converse will take place when the mercury falls, and the end of the glass tube will accurately follow the movement of the surface of the mercury. If the atmospheric pressure increases by a head of one inch of mercury the surface of the mercury falls half an inch in the open limb of the barometer: the pen falls six times that amount, that is, three inches. The instrument has only been constructed in an experimental form, and it was not thought worth while to complicate the instrument by putting an arrangement to prevent the gas escaping into the room. But the leak is so small that with a similar instrument working day and night for some weeks the smell was not perceptible.

I am indebted to Mr Deacon for the design of the pen. It is used by him in his Waste Water Meter and in other self-recording apparatus. These instruments are left for long periods without being touched, the pen drawing the curves in a most satisfactory manner.

In the apparatus exhibited the tracing is made on paper covering a drum rotating about a vertical axis. An American clock movement is fixed to the top of the drum, and the point of rotation of the hands is on its axis. The hour hand projects beyond the side of the drum, and rests against a fixed pin *Q*, and is thus prevented from rotating. Consequently the body of the clock and the drum with it are compelled to turn; the usual condition where the hands rotate and the body remains at rest is reversed.

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(4) *A Verification of a transformation in Elliptic Functions.* By J. CHEVALLIER, B.A.

The author contributed the details of a verification, by direct algebraical methods, of the following theorem:

If $$y = \frac{M^2 X}{Y^3},$$

where $$X = x^5 - 2Ax^4 + Bx^3 - Cx^2 + Dx - E,$$

$$Y = x^2 - Ax + A', \quad \frac{1}{M} = \tfrac{1}{2}(-1 + i\sqrt{19}),$$

$$A = -\tfrac{1}{2}(\sqrt{19} + i), \quad A' = \tfrac{1}{10}(11 + i\sqrt{19}),$$

$$B = \tfrac{1}{2}(25 + 5i\sqrt{19}), \quad C = -\tfrac{1}{2}(\sqrt{19} + 6i),$$

$$D = \tfrac{1}{2}(21 + 9i\sqrt{19}), \quad E = -\tfrac{1}{2}(\sqrt{19} + 11i),$$

then $$4y^3 - g_2 y - g_3 = M^2 (4a^3 - g_2 x - g_3) \left(\frac{dy}{dx}\right)^2$$

where $g_2 = 8$, $g_3 = \sqrt{19}$.

(5) *On some Caves in Portugal.* By H. GADOW, M.A.

In the spring of 1884 I made a zoological excursion through the Algarve, my chief object being to explore the fauna of certain large subterranean caves. According to Murray's *Handbook for Portugal*, and von Maltzan's *Reise nach Algarve*, these caves are situated somewhere near the little village of Alte, nearly in the centre of the Algarve. Whilst travelling northwards from Faro to Loulé I was told of the existence of some other caves in the neighbourhood.

Leaving the high road from Loulé towards Salir, I followed a mountain path leading eastwards over densely wooded very picturesque hills for about two miles, and came to a wild ravine through which dashed the Rio Secco, one of the eastern tributaries of the poisonous Rio Quarteira. According to the geological maps published by the Portuguese Government, the Rio Secco runs through Jurassic and Keuper formations. Its banks are steep, with an incline of 60°, about 200 feet high near the caves, and strewn over with innumerable large and small, but extremely rough boulders, between which grow plenty of locust trees, olives, a few oaks (Q. suber and robur) and shrubs. Alongside, and in the bed of the river a dense and luxurious vegetation of oleander, olives, vines, cork trees and sarsaparilla delights the eye. On the eastern side there is a plateau, nearly flat, with a few olive trees and dwarf palms (Chamaerops humilis). Cultivation of corn has been attempted, but yields very little, because the plateau is

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covered with small rocks, which, although partly cleared away and heaped up here and there, leave only small irregular patches of a few inches of red clay, baked during the rainless summer into the hardness of bricks.

The entrance to the caves, which are two in number, is just below the north-eastern brink of the valley; access to them is easiest from the plateau. The bearings of the caves are as follows:

26° to westward is the village of Salir, distant about 4 miles.

140° to south-east, distant 3 miles, Querença, where the owner lives, a poor peasant of 73 years of age, called Manoel da Silva.

25° to east a small village, called Serra da Corte.

On the western side of the river, opposite the caves, are seen at a distance of one mile several hamlets, distinguished as Corte do Neto and Corte Ximica.

During my visit in April, 1884, the caves did not yield anything but the following bats: Miniopterus schreibersi, Rhinolophus hipposideros, Rh. hippocrepis, Vespertilio murinus; there were no invertebrates visible, nor was there any water in the caves.

The fragment of a recent goat's skull and a few bones of birds made me wish to excavate the caverns, but that was then impossible. It rained in torrents during nearly the whole of April, and there is no inn in the villages, the nearest accommodation being at Loulé, and that is four or five hours off. In short, to explore the caves properly would have required long preparation.

I then (likewise in 1884) visited the *Poço or Buraco dos Mouros* (almost every cavern, ruin, well and castle in the south of Portugal is connected with the Moors by the peasants) near Benafim and Alte. The entrance to the cave lies nearly on the top of an abruptly ending plateau of the Jurassic Rocha da Pena, a mountain nearly 1500 feet above the sea level. We had to lower ourselves for about 15 feet, and reached a funnel about 6 feet high; the floor, covered with blocks, slopes down towards the left for perhaps 50 yards, then comes a sudden turn to the right, which can now—on account of the fallen blocks—be passed only by crawling flat on one's belly, head downwards, because the incline is steep; after having veered round two or three times at right angles towards the right, we suddenly entered an enormous vault with stalagmitic roof and many pillars. The dripping of water is heard in several nooks; except a small pool, holding perhaps a dozen buckets of water, there was no water whatever in this cavern, and that little contained no visible living creatures.

The height of the vault may be from 20 to 25 feet, its diameter 60 feet. There is no outlet or large recess leading anywhere out of this round the dome, except that by which we entered. The

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floor is partly covered with stalagmites, but chiefly by bats' dung, many feet deep, the bats themselves clinging in great numbers to the ceiling. We were a large party, about a dozen, led by our host, the Mayor of Salir, and half a dozen peasants, whom we had at last persuaded to undertake this superstitiously feared exploration. From the church we had provided ourselves with some big wax candles, which gave no light to speak of, and torches made of unravelled grass ropes. These primitive torches could be kept burning only by being rapidly swung round one's head, whereupon they gave a sudden glare of light, smouldered away for a minute, produced a dense mass of smoke, and then went out.

The dome of this cave had never been visited within memory of living men. Von Maltzan could not squeeze his big frame through the narrow passage; but they still tell the story of a priest, who proceeded from the dome into still another chamber, till he reached the source of a large river, when he thought it prudent to return. We all felt rather disappointed; the passage, the chamber, and the large river have all disappeared, and the priest has died long since.

Not far from this large and deep cave there is said to exist another, called Igreja dos Soudos, compared to a church because of its many stalactitic columns.

This year I visited the *caves near Querença* in July. A kind letter of recommendation from His Excellency the Minister of the Colonies, Barboza du Bocage, to the Governador Civil of the Algarve, Joaquim Bivar, in Faro, caused the latter gentleman to request the Administrador do Conselho de Loulé, Joaquim Manoel Adelino Perreira, to facilitate the purpose of my exploration. Six excellent miners, who had recently returned from Spain, were requested to place themselves at my disposal for 18*d.* a day each; moreover, I was allowed the use of the municipal crowbars, pick-axes, hammers and other necessary tools. I had also secured the written permission of Manoel da Silva, the owner of the caves, for exploring the latter, and last not least, Messrs Mason, Bary and Co., the owners of the rich copper mines of São Domingos near Pomorão on the Guadiana, had kindly consented that my friend, Thomas Warden, civil engineer, should accompany me for a time not to exceed a fortnight. My friend being thoroughly master of the Portuguese language, and knowing the ways of the workmen, his practical experience made matters easy and cheerful.

There being no other accommodation, we established ourselves in the antechamber of the south-eastern cave, and, considering that we never left the caves for longer than an hour, to take a bath down below in the Rio Secco, and that we lived there for 7 days and 7 nights, we may well be said to have played the Troglodytes.

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Our provisions consisted of some tinned sardines and tunny fish in oil, and a daily supply from our cave-lord of wine, bread, eggs, pig-butter, i.e. dripping, cucumbers, onions, and once for a treat twenty small potatoes. Our water supply was fetched from a powerful spring in the bed of the river; although apparently good, we deemed it advisable always to run it through one of Lipscombe's small military filters.

With this fare we kept remarkably healthy, not a moment's illness having been felt by any of us. The heat during the daytime was great. From 10 A.M. to 5 P.M. generally from 95° to 101° in the shade and 128° in the sun, but this otherwise intolerable heat was greatly mitigated by a southern breeze, which with great precision set in daily about 10 A.M. and continued with increasing force until 5 P.M. when it left off, sometimes rather suddenly. This refreshing sea-breeze is, I think, caused by the over-heated plains of the Alemtejo. I encountered it likewise on the Serra de Caldeirão, which divides the Algarve from the Alemtejo, but not inside the latter province itself. There, on the treeless and shrubless plains, it was still hotter for us, one day 135° without any wind and all day long on mules' back. During the heat of the day no life is visible; the very insects sit still below the plants and stones or are hidden in the cracks of the bark of the trees. The lizards and geckos have likewise vanished, and the only sound is now and then the shrill noise made by the "Cigarra" (Cicada) in the dense foliage of the locust tree, the only plant which seems really to enjoy the Algravian summer, finding ways and means in the baked hard soil of the hills to keep up the dark green colour of its pretty leaves and to grow its numerous pendant beans. All the other vegetation was shrivelled up or wore the dusky greenish-grey garb of the olive tree. Only along the banks of the river and in the stagnant pools taking its place during the hot season, is there life and rich luxurious vegetation. The pools abound with snakes, fishes, frogs and small tortoises. The river itself was partly dry, but here and there were springs and pools, the water being dammed up by dykes and carefully saved for irrigating the rich groves of vines, melons and Indian corn.

In the antechamber of our cave the temperature ranged from 72° to 80°, at night outside the cave never below 72°. Far inside the cave the thermometer stood permanently at 64°, which agrees very well with the mean annual temperature of the country. Rain does not fall from June to October. During my four weeks' travelling through the Algarve and southern Alemtejo I never saw a cloud, except one morning, when it was hazy and rather close for a few hours. There was likewise not the smallest trace of dew on the hills, which circumstance enabled us to sleep outside the cave, i.e. away from the gnats, mosquitos and fleas, under the

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bright stars, in the calm balmy air, sung to sleep by the frogs down the ravine and by the melancholy call-note of the little scops-owl and the frog-toad Pelodytes. Bats flitted in and out, not much disturbed after all by our work in the daytime.

The formation of the hill, in which the caverns are, seems to be Keuper. As said before, there are two of them; their entrances open below the edge of the valley. They are called Gruta de Solestreira (called by the country folk Sol e Estrella, sun and star); the south-eastern cave of the two, although actually by far the smaller, is distinguished as the Gruta major, because of its wide entrance. The accompanying plan will explain the shape of this cave.

About 25 feet from the entrance is a huge pillar, A. The floor from the entrance onwards to B is nearly horizontal, even and firm, consisting chiefly of red clay, trampled down by the men and goats, who collect there sometimes during the winter for shelter. The long axis of the cave is about 120 ft. long and runs, roughly speaking, from south to north; the width varies from 12 to 20 ft., the height from 5 to 15. At B the cave turns and narrows suddenly; then comes an irregular chamber, in the recess of which are the only stalactites and stalagmites worthy of notice, except those at G. In the eastern wall at C there is, about 3 ft. above the general level, a roundish entrance 3 ft. high, 2 broad; the sides were not rough like all the rest of the cave walls, but peculiarly smooth, looking as if they had been worn off and polished by frequently being crept through. This circumstance made me explore this side-chamber first. The hole leads to a roundish chamber, D, 6 to 7 ft. wide and 5 ft. high; sloping upwards stretches a passage 7 ft. long, 3 ft. high, 2 wide, covered with rubbish and a layer of stalagmite, 2 to 3 inches thick. At E is a sudden turn, gently sloping down towards F; this passage was likewise 7 ft. long, perhaps 3 ft. high and 2 ft. wide, the floor covered with dry, loose, reddish clay. At F is another turn, the passage sloping down sharply and ending in a roundish chamber, 5 ft. wide and 4 ft. high. Its ceiling is formed by a layer one foot thick, with a hole in it, just large enough to squeeze one's head and shoulders through, while standing in the chamber G in an erect position. Your head is then in another chamber, somewhat smaller than that of G, but the greater part filled with beautiful stalactites. We began to dig at D with the following result:

I. One foot of clayish soil and crumbled pieces fallen from the ceiling.

II. Hard stalagmitic shell, 2 to 4 inches thick.

III. Half a foot of dense red clay.

IV. Several teeth of Deer, well preserved, and a considerable

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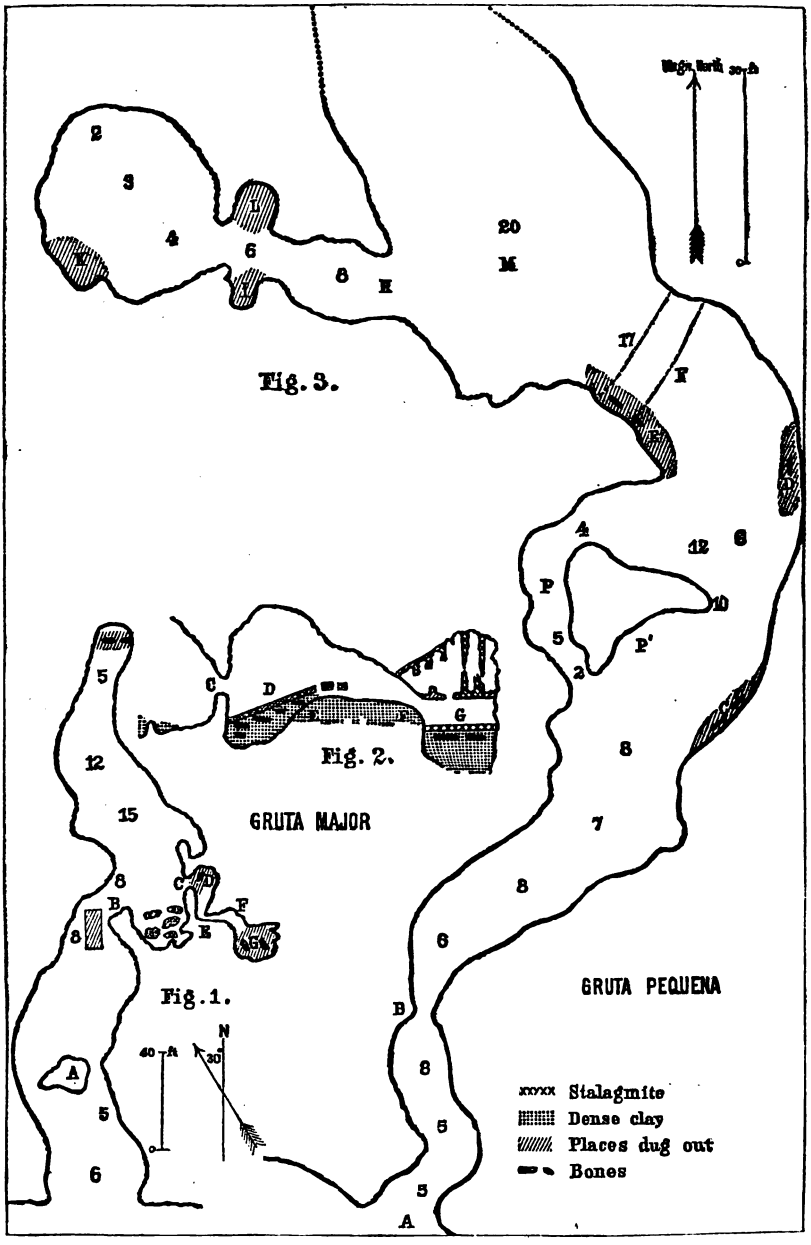

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number of fragments of bones of Deer, Roe-deer and Rabbit, in much damaged condition.

V. 3 feet of red, very dense, somewhat moist clay, without any bones.

VI. Upon this followed the live rock itself.

The end chamber, G:

I. 3 to 4 inches of loose soil, with bats' skeletons, jaw of rabbit, and a few bones of recent birds, mixed up with

II. Pieces of a cracked and partly broken stalagmitic layer. After removal of this layer:

III. Human teeth in good preservation, a small piece of the cranium, some phalanges of fingers and toes, fragments of three radii, of an ulna and of a metacarpal, but everything much damaged. The teeth, several phalanges, and the fragment of one radius belonged to a full-grown person; the other phalanges and radii belonged to a much younger and smaller individual, as is indicated by the size of the bones and by the still separate epiphysis of the distal end of the radii. It was not possible to arrive at a conclusion regarding the original position of the skeletons. Below these bones an instrument made of the cannon-bone of a deer, 8 inches long, 1 broad, the lower end being left untouched so as to form a convenient handle, the other end carefully sharpened off on one side into a cutting broad-bladed dagger[1]. There were also about a dozen small rings, of not more than 0·2 in. diameter, with a round hole in the middle, sometimes 2 or 3 clinging together with their flat sides. They were decidedly not of metal, of greenish or bluish white colour; very brittle, some of them falling to pieces before I could clean these little peculiar objects; some were broken by the workmen,

[1] Baron A. von Huegel has sent me the following remarks on this bone:

This fine specimen, with some celts, flakes, &c. from Portuguese kitchen middens, was presented by Mr Hans Gadow to the Cambridge Museum of Archaeology.

The Solestreira specimen (similar worked bones have been found in various caves both at home and abroad) consists of the metatarsus of a large deer, shaped into an implement, which, to judge from the Cassowary bone "daggers" now made by the Fly River natives of New Guinea, was used not only as a weapon but also for various other purposes, such, for instance, as the digging up of edible roots. Considerable care has been bestowed on the manufacture of this implement which though rough is quite symmetrical. The lower half of the bone has been kept in its natural condition, so as to form the hilt, which would be held in the closed hand, the condyles fitting very comfortably into the palm of the hand. From the remaining portion of the bone the posterior surface, only, has been cut away so as to leave a gradually tapering wedge with a neatly trimmed point, the inner margin of which has been cut and ground into a smooth edge.

Near this specimen were found, as Mr Gadow describes, stone beads and some fragments of human bones. It would be strange indeed were this the entire yield of so large a cave as the Solestreira, and it is but reasonable to suppose that a complete and systematic excavation will bring to light many other objects of prehistoric interest. A. v. H.

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who discovered them and wanted to know what they were. I saved four of them, packed them carefully in wadding and in a small bottle, but this unfortunately was lost together with other things during an accident to the pack mule. There were certainly many of these rings, but we did not discover these tiny things before the excavated soil containing them had been carried from G to D, where there was a little more room for sifting and examining the soil. I have little doubt but that these rings originally formed a necklace and consisted most probably of the mineral described by Dr M. A. Ben-Saude as Ribeirite (nouvelle varieté de la Calaïte); cf. *Compte Rendu de la 9e session du congrès international d'anthropologie et d'archéologie préhistorique*, 1880, Lisbon, pp. 693—696. I have seen such a "grains de colliers," but much larger, found in various caves of Alemtejo and Estremadura, now in the museum of the Academia das Sciencias, in Lisbon.

Together with these human remains were found a few shells of cardium.

IV. Below this layer the soil consisted of compact red clay; we dug out a hole 4 feet deep and then sounded to a depth of 2½ feet more, with the same result. We had thus arrived at a depth of at least 7 feet below the floor of the chamber as we found it. Considering that, as far as we could make out, the stalagmitic shell of the floor of the chamber G was on the same level with that of the principal cave from A to B, our instruments reached about 6 ft. below level AB. It is therefore possible that the true rock-bottom of chamber G is still deeper and may contain other remains.

We then drove a shaft at the furthest end of the principal cave, at A, 5 ft. long, 3 broad: one foot below the surface were a few brittle scraps of bones, but nothing else. We dug 6 ft. deep through red clay and sounded 2 ft. more clay.

Lastly we drove a large shaft at B, 4 ft. deep, and sounded 2½ ft. more, but found the same red clay. Two feet below the surface was charcoal and some broken pottery, which however undoubtedly belonged to an old water-jug as now used in the country. The workmen recognised that sort of pottery at once, and the owner of the cave explained that that portion of the ground had once been levelled to enable him to remove the bats' dung from the inner chamber. Between the stones in the antechamber was discovered a flat and nearly round stone of great hardness, 10 inches in diameter and perhaps 4 inches in thickness, its one surface was hollowed out so as to form a slight depression. The stone looked exactly like those in the Lisbon Museum, which are supposed to have served for grinding corn by hand.

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Gruta pequena.

The north-western cave, called, as I have already stated, Gruta pequena, because of its narrow entrance, is by far the larger of the two. Its long axis runs from south to north and is, measured in the curve, about 250 ft. long. At its northern extremity it leads out to the slope of the plateau through an opening just large enough to admit a small boy. The base or floor of the whole cave is more or less horizontal, but enormous boulders have fallen down from various parts of the ceiling and have rendered many parts very uneven. It is a characteristic feature of this cave that there are several large ledges of stalagmite and of harder and more solid strata projecting horizontally out from the walls. There seems to have been such a stalagmitic layer, at least one foot in thickness, which once covered the whole of the space between C, D, E, G, H. It is highly improbable that this mighty shell had been broken through by man; it must have been destroyed by masses of water; the broken pieces have then been removed by men to enable them to get at the soil below to be used for their fields. The action of the water is plainly visible on the side-walls and on the roof of the large dome, M, all the stalactites having fallen down and the stone having a smooth water-worn appearance.

At B is a narrow entrance, originally only 4 ft. wide and 1 ft. high, closed with a stone slab which about 10 to 15 years ago was removed by the owner and is now used at Querença to serve as a door to his hen-house. At that time, as he told me, "thousands of cartloads of bats' dung, together with the rich clay soil, were removed by the owner as manure for his fields." Between G, E and D the original height of the soil is still visible on the walls, about 5 to 7 ft. above the present level. Whilst digging under the protruding stalagmitic ledge near E he found 3 *sepulturas*, made as he said of cortica (bark of the cork-oak), containing three skeletons, the skulls perfect, even with traces of hair on them, the rest of the bones soft and badly preserved. Although taken to the houses of various peasants, they are now lost. At least I could not ascertain anything definite about their fate, the people being very reluctant to speak on this point. The same applies to "some peculiarly shaped earthenware pots," likewise found there. I do not doubt these accounts, because I heard them independently from various people, and it was only after I had been told about all this by some other peasants that I was able to force some more information out of the owner of the cave.

We dug near E, following the advice of the owner, but found nothing. On the eastern side, under a similar ledge of stalagmite, at D and C we found several much broken bones of man but certainly

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not in their original position, all that ground having been dug previously.

Between F and M is a gigantic vault, about 35 ft. wide and 17 ft. high. All the roof and side-walls of this dome consist of white, water-worn and rather smooth stone, with only a few indications of newly-formed stalactite. Above F, in the ceiling, is a large hole which leads obliquely up towards G. The owner told me, that before the removal of the soil his boy was lifted up there on the shoulders of several men, and that he saw there another slab door like that at B. This of course aroused my curiosity. We tried to climb up on a long pole, but could not get over the edge. Then the six workmen were ordered to build a wall, 12 ft. long and 6 ft. broad at base. After 8 hours' work this wall was about 10 ft. high. We constructed a sort of ladder and succeeded, after removing a large quantity of slippery bats' dung, to enter the upper passage. It is about 4 ft. high, 3 to 2 broad, and ends at G, a very narrow chamber. The ceiling of the latter leads into a vertical funnel, about 18 inches in diameter, and was crammed full of bats. The floor of the chamber G was covered with reddish, soft, but dry soil; digging revealed nothing; there was no outlet whatever, and no slab door. This chamber cannot be far from the surface of the plateau, because several roots of trees had crept through the small cracks in the walls.

On the western side of the dome is a large mass sloping up from H, with several side niches; they contained only red clay. However we did not explore the bottom of the end chamber.

There are plenty of deep and dry recesses in this enormous cavern which have never been disturbed, for instance the passage of P, the space and chamber near K and L. To excavate the floor between M and N would be very difficult owing to the masses of fallen boulders.

Igrejinha dos Mouros.

An hour to the south of these caves, between them and Querença, is a third cave on the plateau, called the "little church of the Moors." The entrance is narrow, rather steep for 10 ft., the bottom horizontal, the passage of the whole cavern only from 2 to 6 ft. wide and 8 to 10 ft. high. Ceiling, walls and floors formed by the most beautiful and large stalactites, which give this cavern the appearance of a small over-ornamented chapel. The people told me that there was there the petrified erect figure of a decapitated Moorish lady. It was a half-transparent stalagmitic pillar 5 ft. high, standing on a bridge of stalagmite in the middle of the cave and bearing indeed a striking resemblance to a woman's figure, and in a yellowish white garment with long and exquisitely arranged folds.

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The general belief of the people is that the Moors, when pressed hard during their expulsion from the country, took refuge in these caves.—This whole cave is about 30 ft. long. Near the further end I found the almost complete half-fossilized skeleton of a goat.

May 10, 1886.

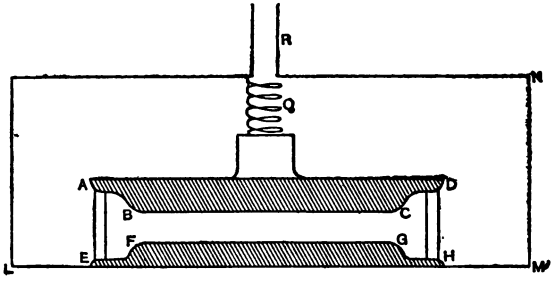
PROF. FOSTER, PRESIDENT, IN THE CHAIR.

The following communications were made to the Society:

(1) *Some experiments on the electric discharge in a uniform electric field, with some theoretical considerations about the passage of electricity through gases.* By Professor J. J. THOMSON.

As the experiments which have hitherto been made on the discharge of electricity through gases have in general been arranged in such a way that it is difficult to calculate what was the state of the electric field before discharge took place, I have thought it might be interesting to make some experiments when the state of the field was accurately known. For this reason I made the discharge take place between two parallel plates separated by a distance which was but small in comparison with their diameters.

Fig. 1.



The arrangement used is represented in fig. 1. *ABCD, EFGH* are two cast-iron plates, the flat portions of which are about 6 centimetres in diameter, and $1\frac{1}{2}$ centimetres apart. They are shaped as in the figure, special care being taken to make the curved parts of the plate smooth and free from places of large curvature; the object of this as well as the peculiar shape of the electrodes is to make the electric field much less intense in those places where it is not uniform than in those places where it is, so that the discharge will take place in the uniform field and not

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in those places where the field is not uniform and difficult to calculate. The surfaces of the plates were worked very true, and some small holes that were left in from the casting were filled up with putty and then coated with gold leaf. The surfaces were so true that though the electrodes were of considerable weight, yet if they were placed in contact they adhered sufficiently to cause the under one to be lifted when the upper one was raised. The plates were maintained at the same distance apart by means of three glass distance-pieces, two of which are shewn at *AE* and *DH*, carefully made of the same length and their ends accurately ground; these were connected together by pieces of glass rod: these distance-pieces were placed in the hollow part of the plates so as to be out of the way of the discharge; the plates were placed in a box, *LMNP*, the side of which was a cylindrical piece of glass and the ends of it brass discs, fastened to the glass with marine glue; into the upper one of these plates a piece of brass tubing, *R*, was soldered in order to permit of the exhaustion of the gas in the vessel; between the top of the box and the upper plate there was a spring, *Q*, which put the two into electrical connection. The spark was produced by means of an induction coil.

The following are the phenomena which occur as the air is gradually exhausted from the box. At the pressure of the atmosphere the spark passes between two points, being evidently determined by some accident which makes the force a little greater at one place than another; at this stage the discharge is very unsteady and skips about from one point of the plates to another: as the pressure diminishes the discharge gradually settles down and remains at one place, and begins to present peculiar features which are represented in the accompanying figures, in all of which the negative electrode is supposed to be at the top. Fig. 2 re-

Fig. 2.

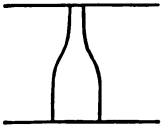


Fig. 3.

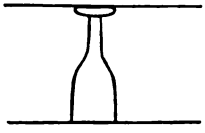
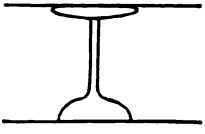


Fig. 4.



presents the discharge when the pressure is that due to about 90 mm. of mercury; it is shaped something like an Indian club with the handle at the negative electrode. As the pressure diminishes, the neck of the club lengthens, the lower part broadens out, and a disc appears at the negative electrode; the appearance at a pressure of about 40 mm. of mercury is represented in fig. 3; the discharge being bluish near the negative electrode, but reddish towards the positive. As the pressure

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falls the disc near the negative electrode broadens out, and the handle of the Indian club lengthens, until the appearance is that represented in fig. 4, which represents the discharge at about 18 mm. of mercury, the difference in colour between the discharge at the positive and negative electrodes being now very marked. As the pressure diminishes the disc at the negative electrode increases in size, until at about 4 mm. this disc appears to constitute the whole of the discharge; it is clearly separated from the negative electrode. I have not been able to detect with any certainty any discharge at the positive electrode, or any glow throughout the tube, and if they exist at this stage they are certainly exceedingly faint. There is a much greater contrast between the bright disc near the negative electrode and the rest of the discharge than between the glow and dark space in a vacuum tube of the ordinary kind. As the pressure diminishes still further, the disc gradually moves further away from the negative electrode, and a decided glow spreads through the vessel; the colour of the discharge keeps changing, and when the pressure sinks below a millimetre it is a pale Cambridge blue. Bright specks also appear over the negative electrode. If an air-break be put in the circuit a curious phenomenon is observed. A glow is distinctly visible between the top of the vessel containing the electrodes and the upper electrode, though these are in metallic connection, and if they acquired the same potential simultaneously, could have no electric field between them. This was only observed when the upper electrode was negative, not when it was positive. The lowest pressure reached with this apparatus was about $\frac{1}{2}$ of a millimetre. At this pressure the disc near the negative, though still observable, was not much brighter than the surrounding glow.

These experiments were repeated, using coal-gas instead of air: very similar results were obtained, except that at high pressures the discharge jumped about more than it did in air. Coal-gas was used because stratifications are usually produced in it with great facility. I never observed any tendency however in the discharge to become striated where the field was uniform, though some small discharges which started from the edges of the positive plates were beautifully striated; on one occasion too a spot of dust had got on the positive electrode in the middle of the uniform discharge, a secondary discharge started from this point, which was very plainly striated, though the main discharge shewed no trace of stratification.

Some experiments were tried with gases which are electrically very weak, such as the vapours of turpentine and alcohol; with these gases I never was able to limit the discharge to a disc near the negative electrode, as was the case when air or coal-gas was

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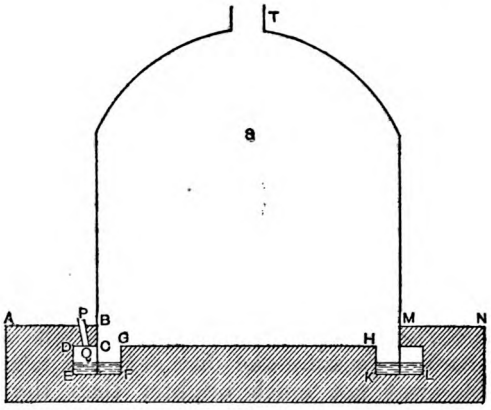
used. With these vapours there was always at low pressures a glow stretching across the space between the electrodes, and though the disc near the negative electrode was distinctly brighter than the rest, I was never able as in air to get the discharge practically confined to the disc. The brightness of the glow was always comparable with that of the disc.

The stratifications of the discharge followed the same law as in coal-gas; in the parts of the discharge where the field was uniform no tendency to stratification could be detected, but any secondary discharge started by some accidental inequality was always distinctly striated.

With the arrangement described above, the pressure could not

Fig. 5.



be reduced below that due to about $\frac{1}{5}$ of a millimetre of mercury, and it was with great difficulty that it was reduced as low as this, so in order to investigate the phenomena at higher exhaustions the parallel plates were put in the apparatus represented in section in fig. 5.

ABCDEF is a brass bed plate with a groove of the shape *CDEFG* cut in it; into this groove the glass vessel *S* which terminates in the tube *T* fits, and is fastened against the side *BC* of the groove by marine glue; mercury is poured into the groove, and the space outside the glass vessel between the mercury and the brass is kept exhausted by a Sprengel pump, which is connected with this space by means of the tube *PQ*. Thus since the pressure in this space can easily be kept by the Sprengel lower than the pressure due to 5 millimetres of mercury, if the depth of the mercury in the groove be greater than 5 millimetres, no air can pass into the vessel, however low the pressure inside may

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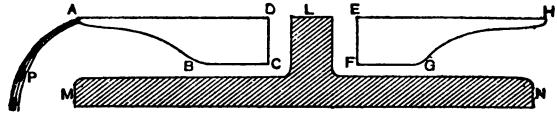
be. The glass tube T was fused on to a Töpler pump. With this arrangement the only joints between the inside of the vessel S and the outside air are mercury ones, and the vessel can readily be exhausted.

The discharge presents the following appearance as the vessel is gradually exhausted. After the pressure gets below the value reached with the first arrangement the glow between the plates gets more and more uniform, until no difference can be perceived in the intensity of the light between the plates; as the pressure diminishes a glow appears above the upper plate, of the kind to be described below, and the intensity of the glow between the plates diminishes; as the pressure falls still lower the glow above the plates increases in intensity while that between the plates diminishes, until at a pressure which I estimated at about $\frac{1}{50}$ of a millimetre there was no glow at all between the plates which were separated by a dark space, while there was a strong glow above the upper plate.

This glow is represented in section in fig. 6. *ABCD, EFGH* represent the glow, which is separated from the plate *LMN* by

Fig. 6.



dark spaces, which are left blank in the figure. The distance of the glow from the plates, that is the width of the dark space, depends upon the intensity of the discharge; by altering the screw of the commutator of the coil the glow could be made to rise and fall in a very striking fashion. The stronger the discharge the smaller seemed the dark space.

From one edge of the glow a bright thread of striated glow, AP, extends, forming apparently the positive part of the discharge, the glow being the negative; this positive part looked like a continuation of the negative, there being no interval that I could see between the glow and the striated discharge. This striated discharge started from the place where the negative glow was farthest from the glass of the bell-jar. It was extremely sensitive to the action of a magnet, the point from which the discharge started being altered by the magnet. The direction in which the discharge moved was along the circumference of the glow, and the direction was determined by the component of the magnetic force along the radius from the centre of the glow to the point where the discharge took place; if the direction of this component was reversed the

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direction of displacement of the glow was reversed. When this component was in one direction the striated discharge was not only deflected but split up into several discharges, there being in this case often 7 or 8 striated discharges proceeding from the negative glow; when the direction of the magnetic force was reversed, the discharge was deflected in the opposite direction, and instead of being split up seemed to be more concentrated than before. This part of the effect seemed to be due to the action of the magnet on the glow; the place where the striated discharge starts is where the glow is furthest from the glass; if the magnetic force by its action on the glow reduces the inequalities in the distance of the edge of the glow from the glass the discharge may start from several places at once, while if it tends to increase the inequalities the glow will be more rigorously confined to one place.

Theoretical considerations about the electric discharges in gases.

In a paper published in the *Philosophical Magazine* for June, 1883, page 427, I gave a theory of the electric discharge in gases, in which the discharge was regarded as the splitting up of some of the molecules of the gas through which the discharge takes place; the energy of the electric field being spent in decomposing these molecules, and finally by the heat given out on the recombination of the dissociated atoms appearing as heat, except in the numerous cases where the gas is permanently decomposed by the spark, when part of the energy of the field remains as potential energy of dissociated gas.

In that paper I did not discuss the difference between the effects observed at the positive and negative electrodes. I think however that the theory is capable of explaining these differences. For we may imagine a molecule of such a kind that the atoms in it would tend to separate when the molecule was moving in one direction in an electric field, say that of the lines of force, but would be pushed nearer together when the molecule was moving in the opposite direction. A molecule of the following kind would possess this property.

Fig. 7.

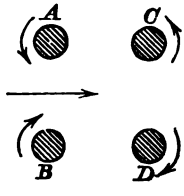
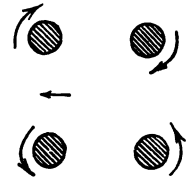


Fig. 8.



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Suppose we have two vortex rings AB and CD of equal strength, whose planes are parallel and whose cores are nearly coincident, they will rotate round each other, the cores remaining at an approximately constant distance apart. Let us suppose that these rings are moving in a fluid which is in motion but in which the distribution of velocity is not uniform; then we know (see a Treatise on the 'Motion of Vortex Rings' by J. J. Thomson, p. 65) that the radii of the rings will alter, and that the alteration will not be affected by reversing the direction of motion of the rings.

Now let us suppose that the radius of AB in consequence of the distribution of velocity in the surrounding fluid increases faster than that of CD, then since the velocity of a ring diminishes as its radius increases the diminution in the velocity of AB will be greater than in that of CD, so that CD will now move faster than AB, the distance between the rings will therefore increase, and if the difference between the velocities is great enough they will ultimately separate. Next let us suppose that the rings are turned round so as to be moving in the opposite direction, as in fig. 8. Then, since the alteration in the radius of either ring is the same after the direction of motion has been reversed; under the same circumstances as before, the radius of the ring AB, which is now in front, will still increase faster than that of the ring CD, which is now in the rear; that is, the diminution in the velocity of the ring in front will be greater than that of the one in the rear, that is, the front ring will move more slowly than the one behind, so that the distance between the rings will diminish and the connection between the atoms in the molecule be made firmer, while in the other case the molecules tended to separate. The only difference between the cases, however, is the direction in which the molecules are moving, so that a molecule of this kind may tend to be decomposed when it is moving in one direction and not when it is moving in the opposite one.

It would, I have no doubt, be possible to give an illustration of this property by taking an ordinary mechanical system and supposing it to be acted on by a proper distribution of forces: the above illustration, however, is sufficient for my purpose, which is to shew that the properties of molecules may be such that they are decomposed when moving in one direction in an electric field but not when moving in the opposite.

Let us trace some of the consequences of supposing that the molecules are decomposed when moving in the direction of the lines of force and not when moving in the opposite direction. If we consider the electric field near the electrodes, this means that at the negative electrode those molecules which are moving towards it are the only ones which have any tendency to be

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decomposed, while at the positive electrode it is only those molecules which are moving away from it which are in this condition.

The consequence of this will be that the molecules will be more easily decomposed at the negative than at the positive electrode. For consider first of all the case of a non-uniform field, when the intensity of the field diminishes as we recede from the electrodes. At the negative electrode those molecules which are approaching the electrode are the ones which tend to get decomposed, and these are going from weak to strong parts of the field, so that the tendency to dissociate gets stronger and stronger, while it keeps getting a better leverage, as it were, for the atoms in the molecule get further and further apart as the molecule moves, and thus the difference in the alteration in their radii would increase even if the field were uniform, but when the field increases in intensity, as the molecule moves on, the effect is still more increased. On the other hand, those molecules at the positive electrode which are likely to be decomposed are those which are moving away from the electrode, and in this case when the intensity of the field is greatest the atoms are nearest together, so that the separating tendency which is the difference in the effects on the atoms is minimized as much as possible; while in the case of the negative electrode, when the tendency to produce a difference was greatest the distance between the molecules was greatest too, so that we see in this case the molecules will dissociate more easily at the negative than at the positive electrode. Again, we must remember that those molecules which are near to the positive electrode and moving away from it, must previously have been approaching the electrode, and that during this time the action of the electric field was to make the atoms come closer together. When the direction of motion of the molecule is reversed by reflection at the positive electrode, the action of the electric field in separating the atoms in the molecule is reversed, so that unless the course of the molecule is extraordinarily unsymmetrical it will be in the same state when it gets away from the electrode as it was before it approached it, and as it was not dissociated in the one case it will not be in the other.

Next let us suppose that the electric field is uniform, as in the experiments described above; then as there is no evidence for any considerable condensation of gas about the electrodes, we shall suppose that the density of the gas is approximately uniform. Since everything is uniform the molecules will dissociate most easily when they are moving for the longest time in the direction of the lines of force. Now according to the vortex atom theory of gases the vortex rings as they approach the planes which form the electrode will expand, and as they expand they move more and more slowly, so that the molecules will be moving for the longest time

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in the same direction in the neighbourhood of the electrodes. And just as in the non-uniform field the molecules will be more likely to dissociate at the negative electrode than at the positive, for at the positive electrode those molecules which are likely to be dissociated are those which are moving away from the electrode, but they must previously have been approaching it, during which time they were being pushed nearer and nearer together, so that at this electrode the molecules which have any tendency to be dissociated are those which have been specially prepared to resist this tendency, and as this is not the case at the negative electrodes the molecules will be dissociated most easily at this electrode.

Hence the conclusion we arrive at is, that whether the field be uniform or variable, dissociation is more likely to take place at the negative electrode than at the positive, and that the dissociation is more likely to take place close to the negative electrode than in the body of the gas; though if the field be very strong or the gas very weak the molecules in the body of the gas may get decomposed. Thus in the experiments described above, though in gases which are electrically strong, such as air and coal-gas, the discharge under certain circumstances could be confined to the neighbourhood of the negative electrode, yet in electrically weak gases, such as the vapours of turpentine and alcohol, the gas was under all circumstances (when the pressure was low) decomposed throughout the field, though the greater brightness of the layer near the negative electrode shewed that more gas was decomposed there than in other parts of the field.

There seems too in the case of the discharge through ordinary vacuum tubes considerable direct evidence that there is a considerable amount of decomposition going on near the negative pole, more so than in the rest of the field, for in the first place, the spectrum of the glowing gas surrounding the negative electrode generally shews lines, while the spectrum in the rest of the field is a band spectrum, and line spectra are believed to denote a simpler molecular constitution than band spectra: and secondly, the gas near the negative electrode is hotter than that in other parts of the field.

Let us trace some of the consequences of the gas being decomposed more easily at the negative than at the positive electrode. Since, according to our view, decomposition of the molecules means a spark, it follows that according to this theory a spark ought to pass more easily from a negative than from a positive pole, a result which was long since observed by Faraday (*Experimental Researches*, § 1501).

Again, decomposition, and therefore discharge, takes place the more easily the longer the molecules move continuously in the direction of the lines of force; thus the longer the average time

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the molecule takes to have its velocity reversed, the more likely is it to be decomposed by the electric field. Now this time will depend on the average time between two collisions and the intensity of the force between the molecules: thus, for example, if the molecules acted on each other like two perfectly elastic spheres, that is, if when they collided they interchanged the velocities parallel to the line joining their centres, the mean time taken to reverse the motion of a molecule would be a little less than twice the mean time between two collisions, for the motion of the molecule will be reversed when it meets a molecule moving in the opposite direction, and it is rather more likely to meet one moving in the opposite direction than one moving in the same direction as itself. Again, it will take longer to reverse the direction of motion when the changes brought about by a collision are small than when they are large; so that taking the ordinary view of a collision as due to forces between the molecules, the greater the forces between the molecules the smaller the time taken to reverse the direction of motion. Other things however being the same, it is evident that this time will vary inversely as the number of collisions per unit time. The particles will be moving very much longer in one direction when the gas is rare than when it is dense; so that the rarer the gas the more easily will the molecules be dissociated: this explains why a rare gas is electrically weaker than a dense one. There must however be a certain density, such that if the rarefaction be carried beyond it the electric strength of the gas will increase as the rarefaction increases; for, according to our view, if there were no molecules there would be nothing to discharge the energy: so that when the density is zero the gas has infinite electric strength. This point will depend, among other things, upon the energy required to dissociate a molecule of the gas into atoms, other things being the same; the greater this energy the lower the critical pressure. It will also depend upon the time during which the molecules are moving in the same direction; if we were to shorten this time without altering the density of the gas or the energy required to dissociate a molecule we should raise the critical pressure, for we should hamper the efforts of the rarefaction to diminish the electric strength, while we should not affect the causes which tend to increase the strength. De la Rue and Müller, *Proceedings of the Royal Society*, 1882, have found that the size of the vessel containing the gas has an effect upon the critical pressure. This is just what we should expect if the above view were true, for the size and shape of the tube will, when the pressure gets so low that the distance traversed by the molecule while it is moving continuously in one direction is comparable with the dimensions of the tube, affect the time during which the molecules are

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moving in the direction of the lines of force, and we should expect that any change in the shape of the vessel which diminished the average time the molecules were moving in the direction of the lines of force would raise the critical pressure.

We must remember that the quantity we are comparing the dimensions of the vessel with is not the mean free path of the molecules, which no doubt at the critical pressure is very small compared with the dimensions of the vessel, but the average space taken to reverse the direction of motion, which may be very much greater. For if the force between the molecules be very small, it will require a great many collisions to reverse the direction of motion of the molecules.

The well-known phenomenon that a thin layer of air is relatively electrically stronger than a thick one, might be explained in the same way as we explained the dependence of the critical pressure upon the shape and size of the containing vessel. For consider the case of two parallel planes, when the distance between the planes is comparable with the average space through which the molecules move continuously in one direction, the constraint caused by the planes will have the effect of diminishing this distance, and therefore of diminishing the chance of the molecule being decomposed by the electric field, it will therefore make the layer stronger than an equally thick layer of gas at the same pressure placed in open space.

The same explanation will also apply to an experiment of Hittorf (Wiedemann *Lehre von der Electricität* bd. IV. § 417); in this a sealed glass flask is furnished with two electrodes made of straight pieces of wire, one being put in the neck, the other in the bulb of the flask. When the one in the bulb is negative it glows all over, but when the one in the neck was negative it only glowed over the tip. In this case the electrode is near the glass sides of the flask, so that the average distance the molecule is moving in the direction of the lines of force is diminished by the glass in those parts of the gas away from the tip more than it is in those near the tip, so that the gas away from the tip is electrically stronger than that close to it. We should therefore expect the discharge to take place at the tip, which is exactly what happens; on the other hand, when the electrode is in the bulb the glass is a great distance from the electrode, it can not therefore affect the distance a molecule travels before its direction is reversed: the gas is therefore equally strong all over the electrode, and we should therefore expect, as is the case, the electrode to be covered with glow.

These considerations would seem to have an important application to the case when there is a contraction in the cross section of the discharge tube at some point. If the tube is

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narrowed so that its dimensions in one direction are comparable with the space it takes to reverse the direction of motion of a molecule, then the gas will be electrically stronger in that direction than in one at right angles to it. This will have the effect of confining the discharge to one direction, and so making it more intense in that direction than it would have been if it had been free to spread out equally in all directions: this perhaps will explain Goldstein's result that the narrowing of the tube produces much the same effect as would be produced if there were a secondary negative electrode at the contraction.

If this theory of the effect of the limitation of the space taken to reverse the direction of motion of a molecule be true, we should expect the following results, which as far as I know have not been investigated.

The ratio of the electric strength of a thin layer to a thick one should be greater at low pressures than at high ones, because the ratio of the average distance the molecules travel along the lines of force before their direction of motion is reversed to the distance between the plates is greater at low pressures than at high ones.

The difference between the electric strength of a thin and a thick layer will depend upon the intensity of the force between the molecules, because the greater the intensity of the force the less the space required by a molecule to reverse its direction; but the smaller the force between the molecules, the greater the coefficient of viscosity, so that the difference between a thick and a thin layer ought to depend upon the coefficient of viscosity of the gas.

The motion of the gas itself produced by the electric discharge must produce important effects on the discharge. When the electric discharge passes through a gas, the pressure of the gas increases, this increase however only lasts as long as the spark passes. This was originally observed by De la Rive, it was afterwards re-discovered by De la Rue and Müller, when investigating the discharge produced by their large chloride of silver battery; by keeping the current on they were able to get an increase of pressure in a large bell-jar of about 30 per cent. of the original pressure (*Phil. Trans.* 1880, p. 86). This increase has been attributed by Dr Schuster (*Proceedings of the Royal Society*, 1884), to the decomposition of the molecules of the gas by the discharge, but though I believe some of the increase to have been due to this cause, yet the effect seems to be too large to be altogether explained in this way, for if the molecules were split in two it would require about 50 per cent. of the molecules to be dissociated,

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and as the discharge took place in a large bell-jar between pointed electrodes, it is hardly possible, I think, to imagine that anything like so large a number of molecules were dissociated by the spark. In addition to this, in some experiments on the subject which Prof. Threlfall and myself have been making during the past year, we got effects which were too large to be explained even by supposing that all the molecules were dissociated, and in these experiments only a small fraction of them became luminous. These results seem to point to a kind of explosion taking place at the place where the spark passes, which projects the surrounding gas away from the place of explosion. We should expect from Faraday and Maxwell's theory of stress in a medium that something of this kind should take place. For according to this theory there is a tension along the lines of force and a pressure at right angles to them; this distribution of stress producing when the field is steady equilibrium at a place where there is no electrification. When however the electric field disappears at one place and not at another the stresses will no longer be in equilibrium, and since the original state was that of tensions along the lines of force, the effect produced by the disappearance of this stress from some part of the field will be much the same as if there was an explosion at the place of discharge, at least as far as the motion of the gas parallel to the line of force is concerned. This effect too will be increased by the decomposition of the molecules which takes place when the spark passes, because this decomposition produces a sudden increase of pressure. For these reasons we conclude that there is a violent projection of the molecules parallel to the lines of force from the neighbourhood of the place where the electric field is discharged.

Let us now consider what takes place when a spark passes through a gas at a moderately high pressure. The electric strength of the gas will break down in the neighbourhood of the negative electrode through some of the molecules there being decomposed. Some of these dissociated atoms will get projected away from the negative electrode, parallel to the lines of force; after going for a short distance, the length of which depends on the density of the gas, they will recombine, giving out heat in so doing; this warms the neighbouring molecules, they are then more easily dissociated, that is they are electrically weaker; the spark passes through them, and the same process is repeated until the spark reaches the positive electrode.

The explosion which takes place when the electric field gets discharged is perhaps one of the reasons why polarization is not produced by the passage of an electric spark through a gas, as it is when an electric current passes through an electrolyte, though according to our view, the process in both cases is very much the

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same. The polarization in an electrolyte is due to the presence of the products of decomposition on the electrodes; if we had any mechanical appliance to wipe them off as fast as they were formed, or if we could prevent them from settling on the electrodes, we should get rid of polarization. Now this is just what is done by the violent explosion which takes place when the spark passes, the explosion drives the products of the decomposition violently about, and prevents them from settling on the electrodes, so that in this case we should not expect any polarization.

The only assumption we have made about the electric field is, that in it there is polarisation of the motion of the ether. Now the motion of the ether must depend to a certain extent upon the motion of the molecules moving about in it. This is very clearly seen in the case of the vortex ring theory of matter, and would seem to be a necessary consequence of any conceivable theory of the relation of matter and the surrounding ether. Thus whenever we have a polarization of the molecules we should expect a polarization of this motion of the ether. Now though it does not follow that the polarization of the ether is always accompanied by electrostatic effects;—the existence of permanent magnets with presumably a polarized arrangement of molecules, and therefore a polarized arrangement of the ether surrounding them, without any corresponding electrostatic phenomenon, seems to shew that it is not—yet if we consider the system of polarized molecules with which we are acquainted, we shall find that in nearly every case they are accompanied by electric phenomena. The property of exhibiting pyro-electric phenomena which all crystals seem to possess, shews, according to Sir WilliamThomson's view of the phenomenon, that in crystals the polarized state of the molecules is accompanied by electric polarization.

Again, the Thomson effect in thermo-electricity i.e. the production of electric effects by the polarized motion of the molecules of a body consequent upon differences of temperature in the body, shews that in this case the polarization of the molecules is accompanied by electrical phenomena. The production of currents in a solution of varying strength, the phenomenon of galvanic polarization, the electrification of bodies by strain, the effects produced in thermo-electric circuits by inequalities in the strain or state of magnetisation, are only some among many of the instances when a polarization of the molecules is accompanied by an electrical effect.

These considerations would seem to have an application to the electric discharge at low pressures. Let us suppose that the electrodes are plates, as in the experiments described above, and suppose the pressure is so low that the discharge spreads to a

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considerable extent over the surface. Then in the gas between the electrodes we have, when the discharge has just commenced next to the negative electrode, a region where the electric field has been discharged, and beyond this a region where the molecules have not been dissociated but into which a few atoms from the dissociated molecules have been projected by the explosion consequent upon the electrical discharge. Now in the region next to the negative electrode the molecules are in a polarized state, since those molecules which were moving in a definite direction have been dissociated. We may expect this polarization of the molecules to produce an electric field, the effect of which on the field, away from the negative electrode, is of an opposite character to that of the original field which produced the electric discharge; if it produced a field of the same kind in this region there would be instability.

Thus the field outside this layer will be less intense than the original field inside it, or than it was outside before the layer of gas next the negative electrode broke down, so that if it did not break down at first, it is not likely to do so afterwards. This would evidently have a tendency to confine the discharge to the neighbourhood of the negative electrode, which was a prominent feature in our experiments with air and coal gas.

Actual measurements by Hittorf and others of the electric force in a discharge tube of the ordinary kind have shewn that this force is very much less outside the first luminous patch which we meet with as we travel from the negative electrode, than it is between this bright patch and that electrode, and this being so it is easy to understand why the discharge should be greatest near the negative electrode.

We shall now go on to apply the views we have been discussing to the case of the electrical discharge through a gas at a low pressure, say less than 2 mm. of mercury. Before doing so however it may be convenient to describe the phenomena observed when the discharge passes through a tube filled with such a gas. Suppose we start from the negative electrode; in contact with this we have a bright glow whose spectrum does not bear much relation to the gas in the tube, it is presumably due to gas absorbed by the electrode or by glowing matter from the electrode itself. Next to this we have a dark space whose dimensions depend upon the density of the gas, getting larger as the density diminishes; next to this we have a luminous mass of gas whose spectrum shews bright lines and whose dimensions depend upon the shape and size of the electrodes and the density of the gas; next to this we have a non-luminous space, for which the name dark interval has been proposed by Dr Schuster; after this we have a glow often striated, reaching up to and touching the positive

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electrode. At very low pressures there are luminous rays which start from the negative electrode in a direction nearly at right angles to it, and travel on in this direction until they meet the glass tube which fluoresces brilliantly at the places where the rays strike the glass.

Let us now consider from our point of view what happens in the tube. The gas near the negative electrode gets decomposed, and the explosion which occurs when the decomposition takes place drives the dissociated atoms about, these after going for some distance recombine, giving out as is usual when gases combine light and heat. The dark space corresponds to the region when the dissociated atoms are moving about, the luminous glow to the space within which they are recombining. The distance through which the molecules must travel before they recombine must evidently depend upon the density of the gas, for the smaller the density the fewer atoms will a given atom meet with in a given distance, and so the fewer chances it has of entering again into combination. Again, the number of collisions which on an average an atom must make before it recombines will depend on the nature of the gas—if the atoms of the gas have a strong affinity for each other fewer collisions will be needed than if the affinity is weak, so that the dimensions of the dark space will be a smaller multiple of the free path in the first case than in the second. Crookes found that the dimensions of the dark space in carbonic acid gas was a smaller multiple of the free path than that in hydrogen. We must however remember that the free path we are considering is not the free path of the molecules but that of the dissociated atoms, and about this we have no direct evidence; again, the disturbance produced by the discharge will no doubt influence the free paths, so that we cannot expect to be able to calculate with any accuracy the dimensions of the dark space from the length of the free path.

The electromotive intensity outside the negative glow is much smaller than that inside, and the molecules are not so easily decomposed as they are near the negative electrode, so that in a region outside the negative glow we have no decomposition of the molecules at all; this region would correspond to the dark interval.

Let us now consider what takes place at the positive electrode: the molecules there are according to our view not so easily decomposed by the electric field as those near the negative electrode, and it may be that, as in our experiments with the parallel plates, we can arrange matters so that practically all the discharge takes place at the negative electrode; in general however the conditions will not be such that the phenomena at the two electrodes are contrasted so much, there will be more molecules decomposed at

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the negative electrode than at the positive, but there will still however be some decomposed at the positive. The molecules which are decomposed at the positive electrode are those moving away from it, for the molecules which are dissociated are those which are moving along the lines of force, and the lines of force point away from the positive electrode. Since the molecules will have to move some distance before they are decomposed, the decomposition will take place some little way from the positive electrode; the explosion produced by the discharge will drive the dissociated molecules in both directions, so that some will be driven towards the positive electrode and will recombine them, so that if the distance which a molecule has to travel before it is decomposed be not too great, the luminous glow will extend right up to the positive electrode. If the field be uniform this glow will stretch away from the positive electrode, and we shall have the glow which we observed in the experiments with parallel plates when we used electrically weak gases.

The very interesting experiments of Dr Schuster on the electric discharge through mercury vapour which is monatomic, and therefore cannot be dissociated, shews that though in this case the electric discharge presents quite different characteristics yet it still exists. The electric field is still discharged, though with great difficulty, so that there must be other means of electric discharge besides that of the decomposition of the molecules.

What is essential for the electric discharge is that there should be some means of dissipating the energy of the electric field; the decomposition of the molecules is one way, and we have seen that there is considerable evidence that part at least of the energy in the electric field is dissipated through this channel, the fact that the discharge takes place through mercury vapour shews that there must be other means of dissipating the energy. The one that seems most obviously suggested by what is perhaps the most striking feature in the discharge, the luminosity, is that the energy is dissipated by radiation. We shall consider some of the consequences of supposing that this is the case. When a system of molecules is acted on by any system of forces, the kinetic theory of gases tells us that the distribution of kinetic energy among the molecules is exactly the same as if no forces acted upon the molecules provided the temperature is the same. So that the only effect which could be produced by the application of a system of forces to a collection of molecules is one that would shew itself as a rise in temperature, the distribution of kinetic energy being a function merely of the temperature and not of the applied forces. Now the radiation presumably depends upon the distribution of

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kinetic energy, so that the radiation of a system of molecules is not altered by the application of a system of forces, provided the temperature be not altered; but the temperature of the molecules in the electric field is only altered slightly, if at all, before discharge takes place, so that the radiation from the molecules can only be slightly affected by the electric field, and there will only be very little if any additional loss from radiation.

The above results however are only true when the applied forces are not intense enough to produce in the intervals between two or at most a small number of collisions a velocity in a coordinate of any type, which is large compared with the average value of that velocity at the temperature of the gas.

When the forces are intense enough to do this in a coordinate of any type it is obvious that the energy of this type will be radiated away enormously more quickly than it would have been if the same amount of energy had been acquired by the molecule so slowly that it had had time to get distributed by collisions among other molecules. Let us apply these considerations to the electric field.

When the intensity of the field is small, the increase in temperature produced by it must be very small, otherwise it would have been observed, hence the energy lost by radiation will be very small, and the gas will insulate almost perfectly. When, however, the field gets so strong that in a space comparable with the free path of a molecule it is able to generate in some coordinate a velocity which is large compared with the average for that temperature; then since the molecule will have to make many collisions before this coordinate is reduced to its average value, and since it is losing energy by radiation all this time, there will be a considerable dissipation of energy, and therefore an electrical discharge. A numerical calculation will shew that enough energy to discharge the electric field might easily be lost by radiation.

In air, at the atmospheric pressure, the maximum electrostatic energy which can exist without there being an electric discharge, is about 500 ergs per c.c., now the energy required to raise the temperature of 1 c.c. of air at the atmospheric pressure 1 degree is about 15000 ergs, and the temperature of the gas would not have to be raised very much for it to lose its heat at the rate of 1 degree per minute; so that if in the electric field the molecules on the average radiated $\frac{1}{30}$ as much as the gas at this temperature the electric energy in the field would be dissipated in one minute. At lower pressures the radiation would not have to be nearly so much, for until we get near to the critical pressure, the electro-motive intensity which the dielectric can stand decreases almost as fast as the pressure, and as the energy in the

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field varies as the square of the electro-motive intensity, the ratio of the greatest possible electro-static energy to the amount of energy required to raise the temperature of the gas 1 per cent. will diminish very rapidly with the density, so that the capacity of radiation to dissipate the electrical energy will increase as the pressure diminishes.

The discharge would also take place more readily at low pressures than at high ones, for at low pressures the free path is greater, and therefore the electric forces have a better chance of producing that abnormally great velocity which is necessary for radiation, and in addition, the collisions are not so frequent; so that when it has got it, it does not run so great a chance of losing it by collision with other molecules.

The discharge by radiation will be much more continuous than that by the decomposition of the molecules, and probably a much larger number of molecules will participate in it.

The decay in the energy in the medium will destroy the equilibrium of the stresses, which according to Faraday and Maxwell, exist in the medium, but as the decay is much slower than if the energy were dissipated by the decomposition of the molecules, the motion of the dielectric which the disappearance of the stress produces will be much less violent.

Whether the discharge of the electric energy takes place by radiation or by the decomposition of the molecules will evidently depend very much upon the nature of the gas; if the gas is one whose particles are easily dissociated, then we should expect the discharge to take place chiefly by the decomposition of the molecules; if, however, the molecules are dissociated with great difficulty then we shall expect the discharge to take place chiefly by radiation. From what we have said, it will be seen that the discharge is more likely to take place by decomposition at the negative than at the positive electrode; so that we may have a case where the discharge is by one means at one electrode and by the other at the other electrode. This I think may be the reason why stratifications are produced more easily in the electrically weak gases, such as the vapours of turpentine and alcohol, than in those which are electrically stronger, for according to our view of striae, their formation depends upon the decomposition of the molecule, and at the positive electrode the discharge does not take place so easily by this means as it does at the negative; so there would be a greater tendency for the discharge to take place in the other way, whatever that way may be, and the formation of the striae would be much facilitated if the molecules of the gas were easily decomposed, as gases of complicated chemical constitution usually are.

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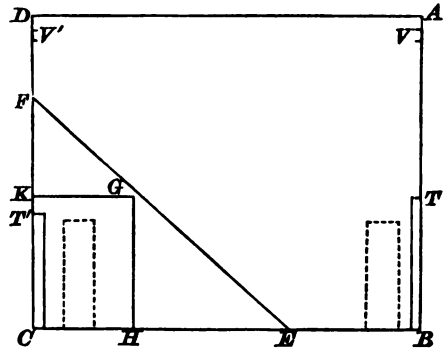
(2) *On an experiment in Ventilation.* By W. N. SHAW, M.A.

This paper results from an attempt to improve the ventilation of the Biological Lecture-Room of the New Museums in order to make the room serviceable for lectures to large classes for consecutive hours.

The room is of the ordinary lecture-room shape, a rectangular room partly filled by raised seats so that the longitudinal section would be somewhat as represented in Fig. 1; *A, B, C, D* being the

Fig. 1.



rectangular outline and *E, F* the slope of the line of seats. There are two doors in the North wall near the corners *B, C*, that near *C* opening into a passage under the higher seats, represented by *HGK*. There is another door in the East wall near the corner diagonally opposite to the door *C*. The room is heated by a row of hot-water pipes cased in wood running along the floor on the South side, immediately under the row of windows in the same wall by which the room is lighted.

The means of ventilation hitherto provided consisted of two Tobin inlets *T, T'* in opposite corners, and two exhaust ventilators *V, V'*.

The Tobin *T* in the corner *B* of the North and East walls is one square foot in area and is supplied by air from the space underneath the floor of the room, communicating with the outside by a number of open grids. In this space there are numerous hot-water pipe connections so that the air delivered by the Tobin is warmed, and enters the room at a temperature of about 80° F.

The other Tobin, in the opposite corner where the passage turns to enter the room, is about 12″ × 13″ in area and com-

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municates directly with the outside air, and consequently delivers cold air.

The ventilators V, V' are not far from the middle of the two end walls and close to the ceiling.

The accepted theory of ventilation upon which this arrangement is based is that a source of heat in the room warms the air, which rises to the ceiling and passes out by the flues of the ventilators V, V'. The sources of heat in the room in question may be taken to be the lecture class. The air which they have used will according to this theory rise to the ceiling and be drawn out by the ventilators as desired, its place being supplied by fresh air delivered by the Tobins.

Other sources of heat which should produce a like effect are the hot-water pipes; these, by the simple application of the theory, should cause the air near them to rise and pass out by the ventilators in like manner. That this is not the case is clearly shewn by the fact that the room is rather more than comfortably warm; but the row of windows above them would produce an opposing downward current and may very easily account for the diffusion of the hot air produced by these pipes.

The general theory of ventilation has been illustrated by various lecture-room experiments and I found some additional ones in endeavouring to trace the course of the air which the Tobins T, T' supplied to the room. For this purpose I used a gold-beater's skin balloon filled with gas only to such an extent as to make it as nearly as possible of the same density as the air of the room. On placing this over the Tobin T, which delivered warm air, it rose at once to the ceiling and very slowly made its way along there, so that we may conclude that this air passed at once to the ceiling and there formed a warm layer. The cold air from the Tobin T' on the other hand carried the balloon downwards to the floor at once and it then passed along the passage towards the door at C. If placed close in the corner over the Tobin, the current of incoming air raised it about 3 feet and then carried it over, but the impression conveyed was that the air behaved very much like water flowing over the top of the Tobin.

I did not find it possible by means of the balloon to trace the course of the air far along the ceiling or floor respectively. The only other additional current was one from the passage up the stairs leading into the room. At the ceiling over the seats the air seemed almost motionless but the fact that the ventilators were active was amply evidenced by the balloon being firmly held against them when brought near to them.

It would appear therefore that, whatever happens to the vitiated air, in actual practice and in accordance with the usual

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theory, the warm air supplied by the Tobin *T* does pass to the ceiling and that the ventilator *V* derived a large part of its supply directly from that source, and that this Tobin was practically useless as a means for renewing the air of the room.

The theory of ventilation is no doubt applicable in cases like the one mentioned, in which the source of heat heats sufficient air to form a steady convection current, such as would be generated by the hot air in the shaft of the Tobin *T*, or by means of a gas flame, but with some hesitation I offer the following reasons for regarding the diffusion of air surrounding persons in a room as a much more complex phenomenon; which in a case like the one we are considering would result not in a transference of the bad air to form a layer along the ceiling leaving fresh air beneath, but in a local circulation i.e. passing upwards for some short distance and then sinking again over the space not occupied by the audience.

The air which is respired does not rise upwards without mixing with the surrounding. We may form some idea of the rate at which mixing goes on from the fact, that, in breathing into a cold room a cloud is formed which disappears almost instantly, shewing that the air has become sufficiently mixed to take it above the saturation point by the time the expiration is completed. Assuming that 30 cubic inches are expired and that these contain 4 p. c. of carbonic acid and are saturated with moisture at the temperature of 35° C. and breathed into an atmosphere of 15° C. I find that in order to reduce it to air which is just saturated its volume must have increased to nearly twice its original value. Its temperature would then be 28° C. and its density ·96 of that of the surrounding atmosphere, thus giving a lifting force on each gramme of air of about 3 centigrammes; this would be very rapidly diminished by further mixing. Moreover the moist air would rapidly radiate heat through the drier surrounding air and approach very nearly to its temperature, so that before it had been raised any considerable distance its lifting force, due to difference of temperature, would have become inappreciable. It would have become generally diffused at some small height above the people at the same temperature as the surrounding air and would be loaded with carbonic acid, which would render it heavier, and thus unless mechanically removed it would tend to sink and replace the colder air taken from below to supply its place when it originally rose.

In the Biological lecture-room the mechanical effects of the ventilators and the cold Tobin *T'* would interfere to some extent with the establishment of the circulation in question, and the effect may not be so readily perceived as in some other instances, such as the lecture-room of the Cavendish Laboratory, which furnishes a very good example. In that lecture-room a dozen

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persons sitting on the raised seats under which are hot-water pipes are sufficient to make the atmosphere behind the lecture-table quite appreciably foul. I have in contemplation some experiments to shew the existence of this circulation but I have not yet had time to complete them.

That hot air projected into cold does not rise with any great rapidity is further illustrated by the steam which issues from locomotives, which, although very much heated when ejected, lies on a still day very little above the train level, and contrasts remarkably in its behaviour with the vortex ring of steam that is sometimes seen and which penetrates the air with much greater rapidity.

These considerations tend to shew that the circulation of air in a room cannot be regarded as a simple case of convection current; probably in many cases the result depends more upon diffusion than convection.

The ventilation of the lecture-room might be considered to have been faulty in these respects:—

(1) The hot-air Tobin T delivered its air direct to the ceiling and there formed a layer; a large part of it probably escaped directly by the ventilator V.

(2) The row of hot pipes also produced an upward current, warming some of the air supplied by the cold Tobin T' and sending it direct to the ceiling. This as stated would be to a certain extent remedied by local circulation produced by the windows.

(3) The foul air immediately over the audience probably formed a circulation of its own; part indeed escaping by the ventilator V', and the rest falling upon the lecturer.

There are two acknowledged principles of efficient ventilation, the first to supply fresh air in such a way that the position it will take up is known, and in such quantity that it must extend rapidly enough over the whole room, and then to let the foul air find its way out; the second to remove the foul air by means of exhaust pipes drawing air from where it is known to be foul and leaving the fresh air to find its way in. In altering the ventilation of the room in question I have endeavoured to secure the advantages of both these methods in the following manner.

The Tobin T forms a layer of hot air at the ceiling. I have prevented this escaping by covering the ventilator V by a wooden shaft attached to the wall which has an opening $24'' \times 14''$ inclined at an angle of 45° to the wall about 6 feet above the floor. The layer of hot fresh air must therefore gradually thicken and in time fill the room (neglecting that part of it which will escape unused by the ventilator V'). This is the application of the first method.

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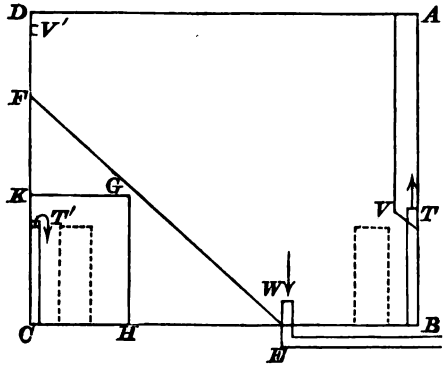
To secure the advantages of the second method I have laid an open tube 24″ × 12″, one opening of which is horizontal and is on the level of the rail of the partition between the lecturer and the audience and just behind that partition, the other opening is in the boiler room where are the two furnaces of the hot-water apparatus of that part of the building. By closing the door of that room and measuring the amount of air which went through the window to supply the fires I had calculated that I could by means of this tube extract 18,000 cubic feet of air per hour from a part of the lecture-room where the air was certainly impure. I had intended to increase the hot air inlets by adding a Tobin between two windows on the south side connected with one part of the case of the hot-water pipes, providing the opening to the outside air in another part of the same case, but this has not yet been done.

The present arrangement of exhaust and supply Tobins may therefore now be represented by Fig. 2.

Fig. 2.



T and *T*′ are the old Tobins, *V* the new position of the ventilator in the east wall, *V*′ the ventilator in the west wall as before. *W* the new tube connected with the furnace room.

I have made some measurements of the amount of air which passes through the various openings, first with the new shaft open and secondly when it is closed. The results are given in the following table. The cubical content of the room is about 36000 cubic feet.

The measurements were taken by means of a Casella wind-gauge. They are of course very rough and especially so in the case of the inlets, the rate for which varied very considerably with the wind. It will be seen that the new shaft does very much what

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it was expected to do in the way of extracting air, and that without interfering with the action of the ventilators, which still behave as exhausts.

	Inlets.			Outlets.			
	T 12″ × 12″	*T′* 12″ × 13″	Total.	*W* 18″ × 12″	*V* 24″ × 14″	*V′* 22″ × 10″	Total.
Air passing in cu. ft. per hr. *W.* open.	5480	13200	18680	18960	13080	9000	41040
Air passing in cu. ft. per hr. *W.* shut.	4920	11580	16500	0	14520	8280	22800

Besides the Tobin inlets I have found that when the shaft *W* is open, there is a supply of air from both the doors at *B* and *C*, but an outlet to the door in the corner opposite to *C*. When the shaft is closed there seems to be a reversal of the currents, under the doors at *B* and *C*, so that in the one case the staircase outside is fed by the lecture-room and in the other the reverse is the case. This may account for some of the very large difference between the exhaust and the inlet measurements.

The effect upon the atmosphere of the room is of course not to be expressed by measurement; the reports however of some of those who use it seem to shew that the experiment has to a certain extent succeeded.

In the report of the Parliamentary Commission on warming and ventilation of dwellings (1857), Roscoe gives the amount of air required per person for continuous occupation of a room as from 15 to 20 cubic feet per minute. This would require for a lecture-room, occupied by a class of 100 persons, from 90,000 to 120,000 cubic feet per hour, so that the exhaust at present operative of 41,000 cubic feet is still deficient.

(3) *On the series for* e^x, $\log_e(1 \pm x)$, $(1 + x)^m$. By M. J. M. Hill, M.A.

The object of this paper is to give a proof of the series for $\log_e(1 \pm x)$, which does not employ the method of indeterminate coefficients nor the method of limits.

The necessary work nearly completes a proof of the Binomial Theorem which does not involve the principle of permanence of equivalent forms.

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The whole investigation being based on the Exponential Theorem, the writer has developed a proof of this theorem distinctly indicated, but not fully set forth, by Sir W. R. Hamilton in his Theory of Conjugate Functions or Algebraic Couples in the Transactions of the Royal Irish Academy for 1837 (pp. 411—412). As the writer has not seen this proof in any text-book, the Cambridge Philosophical Society may consider it worth printing.

The Binomial and Multinomial Theorems for positive integral indices will be assumed to be known.

Art. 1. The proof of the Exponential Theorem proceeds very nearly after the manner of Euler's proof of the Binomial Theorem, except that here the equation

$$f(m) \times f(n) = f(m+n)$$

where $$f(m) = 1 + m + \frac{m^2}{2!} + \ldots + \frac{m^r}{r!} + \ldots$$

is directly proved.

Art. 2. The identity of the series

$$1 + n\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) + \frac{n^2}{2!}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^2 + \ldots + \frac{n^r}{r!}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^r + \ldots$$

and the series

$$1 + nx + \frac{n(n+1)}{2!}x^2 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + \ldots$$

is established.

Art. 3. Putting in the identity of the last article, $n = 1$; using the Exponential Theorem, and the known expression for the sum of a Geometrical Progression, it is shewn that

$$e^{\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} \ldots\right)} = \frac{1}{1-x}$$

whence $$\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = -\log_e(1-x),$$

which is the logarithmic expansion.

Art. 4. Making use of the Exponential Theorem and the result of the last article, the first series in Art. 2 may be written

$$e^{n\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)} = e^{-n\log_e(1-x)} = (1-x)^{-n};$$

$$\therefore (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{2!}x^r + \ldots$$

which gives the Binomial Theorem.

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Art. 1. The Exponential Theorem.

(α) Let $f(m)$ denote the series $1 + m + \frac{m^2}{2!} + \ldots + \frac{m^r}{r!} + \ldots$

It is required to show that whatever m and n may be

$$f(m) \times f(n) = f(m+n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

Let the coefficient of $m^r n^s$ be calculated on both sides.

On the left hand it is $\frac{1}{r!} \cdot \frac{1}{s!}$.

On the right hand the term $m^r n^s$ can only occur in the term $\frac{(m+n)^{r+s}}{r+s!}$. Its coefficient is therefore $\frac{1}{r+s!} \cdot \frac{r+s!}{r!\ s!}$.

The coefficient of any term $m^r n^s$ being the same on both sides, the equation is demonstrated.

(β) Now suppose x a positive integer, then

$[f(1)]^x = f(1) \times f(1) \times f(1) \times \ldots$ until there are x factors

$= f(x)$, by repeated use of the equation (A).

(γ) Next let x be a positive fraction $\frac{p}{q}$, where p and q are positive integers;

$$\therefore f(x) = f\left(\frac{p}{q}\right).$$

But $f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times \ldots$ until there are q factors,

$= f(p)$, by repeated use of the equation (A),

$= [f(1)]^p$, by the previous case since p is a positive integer;

$$\therefore \left[f\left(\frac{p}{q}\right)\right]^q = [f(1)]^p,$$

$$\therefore f\left(\frac{p}{q}\right) = [f(1)]^{\frac{p}{q}},$$

$$\therefore f(x) = [f(1)]^x.$$

(δ) Lastly, let x be negative and $= -n$, so that n is positive.

Then $f(-n) \times f(n) = f(0)$, by equation (A),

$$= 1,$$

$$\therefore f(-n) = \frac{1}{f(n)};$$

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$$\therefore f(x) = f(-n) = \frac{1}{f(n)} = \frac{1}{[f(1)]^n}, \text{ by the previous case}$$
$$= [f(1)]^{-n}, \text{ by the definition of a negative index}$$
$$= [f(1)]^x.$$

Hence whatever x may be

$$f(x) = [f(1)]^x.$$

But $f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$ which is usually denoted by the symbol e,

$$\therefore f(x) = e^x,$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$$

whatever x may be.

The series is convergent so long as x is finite.

Art. 2. It is required to establish the identity of the series

$$1 + \sum_{r=1}^{r=\infty} \frac{n^r}{r!} \left(\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^s}{s} + \ldots\right)^r,$$

and

$$1 + \sum_{r=1}^{r=\infty} \frac{n(n+1)\ldots(n+r-1)}{r!} x^r;$$

the summations being effected for all positive integral values of r.

The first series is $1 + n\left(\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^r}{r} + \ldots\right)$

$$+ \frac{n^2}{2!}\left(\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^r}{r} + \ldots\right)^2$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{n^r}{r!}\left(\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^r}{r} + \ldots\right)^r$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

Let it be written for brevity $c_0 + c_1 x + c_2 x^2 + \ldots + c_r x^r + \ldots$

Then $c_r = n \cdot \frac{1}{r}$

$$+ n^2 \Sigma \frac{1}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} \cdot \frac{1}{1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 2 \end{cases}$$
$$+ n^3 \Sigma \frac{1}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} \cdot \frac{1}{1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 3 \end{cases}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ n^r \Sigma \frac{1}{\alpha_1!\,\alpha_2!\ldots\alpha_r!} \cdot \frac{1}{1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = r. \end{cases}$$

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The Σ in each case denotes summation for all possible solutions of the corresponding equations in $\alpha_1, \alpha_2 \ldots \alpha_r$, in which each of these quantities have any positive integral values between 0 and r inclusive.

Remembering that $r = 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r$,

multiplying both sides of the equation for c_r by

$$\frac{r}{n} = \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{n};$$

$$\frac{rc_r}{n} =$$

$$1 + n\Sigma \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{\alpha_1!\,\alpha_2! \ldots \alpha_r!\, 1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 2 \end{cases}$$

$$+ n^2\Sigma \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{\alpha_1!\,\alpha_2! \ldots \alpha_r!\, 1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 3 \end{cases}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ n^{r-1}\Sigma \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{\alpha_1!\,\alpha_2! \ldots \alpha_r!\, 1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = r. \end{cases}$$

Now the coefficient of n^{r-s-1} is

$$\Sigma \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{\alpha_1!\,\alpha_2! \ldots \alpha_r!\, 1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = r - s, \end{cases}$$

the summation extending to terms corresponding to all possible solutions of the equations in $\alpha_1, \alpha_2 \ldots \alpha_r$, where these quantities have any positive integral values between 0 and r inclusive.

Suppose that in one set of values of the quantities $\alpha_1, \alpha_2 \ldots \alpha_r$, which satisfy the equations, the value of α_t is not zero (the value of any or of all the remaining quantities $\alpha_1 \ldots \alpha_{t-1}, \alpha_{t+1} \ldots \alpha_r$ may be zero), then the corresponding term is

$$\frac{1\alpha_1 + \ldots + (t-1)\,\alpha_{t-1} + t\alpha_t + (t+1)\,\alpha_{t+1} + \ldots + r\alpha_r}{\alpha_1! \ldots \alpha_{t-1}!\,\alpha_t!\,\alpha_{t+1}! \ldots \alpha_r!\, 1^{\alpha_1} \ldots (t-1)^{\alpha_{t-1}} t^{\alpha_t} (t+1)^{\alpha_{t+1}} \ldots r^{\alpha_r}}.$$

This breaks up into the sum of terms, of which

$$\frac{t\alpha_t}{\alpha_1! \ldots \alpha_{t-1}!\,\alpha_t!\,\alpha_{t+1}! \ldots \alpha_r!\, 1^{\alpha_1} \ldots (t-1)^{\alpha_{t-1}} t^{\alpha_t} (t+1)^{\alpha_{t+1}} \ldots r^{\alpha_r}}.$$

is a type; and this may be expressed thus

$$\frac{1}{\alpha_1! \ldots \alpha_{t-1}!\,\alpha_t - 1!\,\alpha_{t+1}! \ldots \alpha_r!\, 1^{\alpha_1} \ldots (t-1)^{\alpha_{t-1}} t^{\alpha_t - 1} (t+1)^{\alpha_{t+1}} \ldots r^{\alpha_r}}.$$

Now putting

$$\alpha_1 = \beta_1, \ldots \alpha_{t-1} = \beta_{t-1},\ \alpha_t - 1 = \beta_t,\ \alpha_{t+1} = \beta_{t+1}, \ldots \alpha_r = \beta_r$$

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this becomes

$$\frac{1}{\beta_1!\ldots\beta_{t-1}!\,\beta_t!\,\beta_{t+1}!\ldots\beta_r!\,1^{\beta_1}\ldots(t-1)^{\beta_{t-1}}t^{\beta_t}(t+1)^{\beta_{t+1}}\ldots r^{\beta_r}},$$

which is of similar form to the terms in the coefficients of the powers of n in the expression for c_r, but here

$$1\beta_1+\ldots+(t-1)\,\beta_{t-1}+t\beta_t+(t+1)\,\beta_{t+1}+\ldots+r\beta_r$$
$$=1\alpha_1+\ldots+(t-1)\,\alpha_{t-1}+t(\alpha_t-1)+(t+1)\,\alpha_{t+1}+\ldots+r\alpha_r=r-t,$$

and
$$\beta_1+\ldots+\beta_{t-1}+\beta_t+\beta_{t+1}+\ldots+\beta_r$$
$$=\alpha_1+\ldots+\alpha_{t-1}+(\alpha_t-1)+\alpha_{t+1}+\ldots+\alpha_r=r-s-1.$$

In these equations t must be $\geqq 1$, and also since none of the β's are negative and one at least must be positive

$$\beta_1+\ldots+\beta_t+\ldots+\beta_r\leqq 1\beta_1+\ldots+t\beta_t+\ldots+r\beta_r,$$
$$\therefore\ r-s-1\leqq r-t,$$
$$\therefore\ t\leqq s+1.$$

Hence
$$1\leqq t\leqq s+1.$$

Hence on breaking up all the terms included in

$$\Sigma\frac{1\alpha_1+2\alpha_2+\ldots+r\alpha_r}{\alpha_1!\,\alpha_2!\ldots\alpha_r!\,1^{\alpha_1}2^{\alpha_2}\ldots r^{\alpha_r}},\ \text{where}\ \begin{cases}1\alpha_1+2\alpha_2+\ldots+r\alpha_r=r,\\ \alpha_1+\alpha_2+\ldots+\alpha_r=r-s,\end{cases}$$

there is obtained the sum

$$\Sigma\frac{1}{\beta_1!\,\beta_2!\ldots\beta_r!\,1^{\beta_1}2^{\beta_2}\ldots r^{\beta_r}},$$

where
$$\begin{cases}1\beta_1+2\beta_2+\ldots+r\beta_r=r-t,\\ \beta_1+\beta_2+\ldots+\beta_r=r-s-1,\end{cases}$$
$$1\leqq t\leqq s+1.$$

And in this last summation terms corresponding to all possible solutions of the equations

$$1\beta_1+2\beta_2+\ldots+r\beta_r=r-t,$$
$$\beta_1+\beta_2+\ldots+\beta_r=r-s-1,$$
$$1\leqq t\leqq s+1,$$

are included.

For take the solution $\beta_1=\gamma_1,\ \beta_2=\gamma_2,\ \ldots\ \beta_t=\gamma_t,\ \ldots\ \beta_r=\gamma_r$, then

$$1\gamma_1+2\gamma_2+\ldots+t\gamma_t+\ldots+r\gamma_r=r-t,$$
$$\gamma_1+\gamma_2+\ldots+\gamma_t+\ldots+\gamma_r=r-s-1,$$

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which can be re-written in one way only in the form

$$1\gamma_1 + 2\gamma_2 + \ldots + t(\gamma_t + 1) + \ldots + r\gamma_r = r,$$
$$\gamma_1 + \gamma_2 + \ldots + (\gamma_t + 1) + \ldots + \gamma_r = r - s;$$

showing that the term corresponding to the solution

$$\beta_1 = \gamma_1,\ \beta_2 = \gamma_2, \ldots \beta_t = \gamma_t, \ldots \beta_r = \gamma_r,$$

can be derived by breaking up that term in

$$\Sigma \frac{1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r}{\alpha_1!\,\alpha_2! \ldots \alpha_r!\, 1^{\alpha_1} 2^{\alpha_2} \ldots r^{\alpha_r}}, \text{ where } \begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = r - s, \end{cases}$$

for which $\alpha_1 = \gamma_1,\ \alpha_2 = \gamma_2, \ldots \alpha_t = \gamma_t + 1, \ldots \alpha_r = \gamma_r$; and from that term only.

Hence the coefficient of n^{r-s-1} in $\frac{rc_r}{n}$ is

$$\Sigma \frac{1}{\beta_1!\,\beta_2! \ldots \beta_r!\, 1^{\beta_1} 2^{\beta_2} \ldots r^{\beta_r}},$$

for all possible solutions of the pairs of equations

$$\beta_1 + \beta_2 + \ldots + \beta_r = r - s - 1$$
and $$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r - 1;$$
or $$\beta_1 + \beta_2 + \ldots + \beta_r = r - s - 1$$
and $$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r - 2;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
or $$\beta_1 + \beta_2 + \ldots + \beta_r = r - s - 1$$
and $$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r - t;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
or $$\beta_1 + \beta_2 + \ldots + \beta_r = r - s - 1$$
and $$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r - s - 1.$$

Now consider the coefficient of n^{r-s-1} in

$$c_0 + c_1 + c_2 + \ldots + c_{r-1}.$$

It consists of parts contained in

$$c_{r-s-1} + c_{r-s} + c_{r-s+1} + \ldots + c_{r-1}.$$

Therefore

(remembering that the coefficient of n^p in c_q is $\Sigma \frac{1}{\alpha_1!\,\alpha_2! \ldots \alpha_q!\, 1^{\alpha_1} 2^{\alpha_2} \ldots q^{\alpha_q}}$ where $\begin{cases} 1\alpha_1 + 2\alpha_2 + \ldots + q\alpha_q = q \\ \alpha_1 + \alpha_2 + \ldots + \alpha_q = p \end{cases}$),

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the coefficient of n^{r-s-1}

in c_{r-s-1} is $\Sigma \dfrac{1}{\alpha_1! \ldots\ldots \alpha_{r-s-1}!} \cdot \dfrac{1}{1^{\alpha_1} \ldots\ldots (r-s-1)^{\alpha_{r-s-1}}}$

where $\begin{cases} 1\alpha_1 + \ldots + (r-s-1)\,\alpha_{r-s-1} = r-s-1 \\ \alpha_1 + \ldots + \alpha_{r-s-1} \qquad\qquad = r-s-1 \end{cases}$;

in c_{r-s} is $\Sigma \dfrac{1}{\alpha_1! \ldots\ldots \alpha_{r-s}!} \cdot \dfrac{1}{1^{\alpha_1} \ldots\ldots (r-s)^{\alpha_{r-s}}}$

where $\begin{cases} 1\alpha_1 + \ldots + (r-s)\,\alpha_{r-s} = r-s \\ \alpha_1 + \ldots + \alpha_{r-s} \qquad\quad = r-s-1 \end{cases}$;

.....

in c_{r-t} is $\Sigma \dfrac{1}{\alpha_1! \ldots\ldots \alpha_{r-t}!} \cdot \dfrac{1}{1^{\alpha_1} \ldots\ldots (r-t)^{\alpha_{r-t}}}$

where $\begin{cases} 1\alpha_1 + \ldots + (r-t)\,\alpha_{r-t} = r-t \\ \alpha_1 + \ldots + \alpha_{r-t} \qquad\quad = r-s-1 \end{cases}$;

.....

in c_{r-1} is $\Sigma \dfrac{1}{\alpha_1! \ldots\ldots \alpha_{r-1}!} \cdot \dfrac{1}{1^{\alpha_1} \ldots\ldots (r-1)^{\alpha_{r-1}}}$

where $\begin{cases} 1\alpha_1 + \ldots + (r-1)\,\alpha_{r-1} = r-1 \\ \alpha_1 + \ldots + \alpha_{r-1} \qquad\quad = r-s-1 \end{cases}$.

Now consider the equations

$$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r-t$$
$$\beta_1 + \beta_2 + \ldots + \beta_r = r-s-1,$$

which written more fully are

$$1\beta_1 + 2\beta_2 + \ldots + (r-t)\,\beta_{r-t} + (r-t+1)\,\beta_{r-t+1} + \ldots + r\beta_r = r-t$$
$$\beta_1 + \beta_2 + \ldots + \beta_{r-t} + \beta_{r-t+1} + \ldots + \beta_r = r-s-1.$$

In these it is impossible that

$$\beta_{r-t+1},\ \beta_{r-t+2},\ \ldots\ \beta_r$$

can be different from zero, for if any one of them were, the left-hand side of the first of these equations would be greater than the right, because none of the β's are negative.

Hence these equations have the same solutions as the equations

$$1\beta_1 + 2\beta_2 + \ldots + (r-t)\,\beta_{r-t} = r-t$$
$$\beta_1 + \beta_2 + \ldots + \beta_{r-t} \qquad = r-s-1$$

(leaving out of account the zero values of $\beta_{r-t+1}, \beta_{r-t+2} \ldots \beta_r$, which in no way affect the terms following the symbol Σ in the previous part of the investigation).

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Hence the equations

$$1\beta_1 + 2\beta_2 + \ldots + r\beta_r = r - t$$
$$\beta_1 + \beta_2 + \ldots + \beta_r = r - s - 1$$

have the same solutions as the equations

$$1\alpha_1 + 2\alpha_2 + \ldots + (r-t)\alpha_{r-t} = r - t$$
$$\alpha_1 + \alpha_2 + \ldots + \alpha_{r-t} = r - s - 1,$$

where $$1 \leqq t \leqq s+1.$$

Hence coefficient of n^{r-t} in $\frac{rc_r}{n}$ is the same as in

$$c_0 + c_1 + c_2 + \ldots + c_{r-1}.$$

Hence coefficients of all powers of n are the same in $\frac{rc_r}{n}$ and

$$c_0 + c_1 + c_2 + \ldots + c_{r-1};$$

$$\therefore \quad \frac{rc_r}{n} = c_0 + c_1 + c_2 + \ldots + c_{r-1},$$

whence $$\frac{(r-1)c_{r-1}}{n} = c_0 + c_1 + c_2 + \ldots + c_{r-2};$$

$$\therefore \quad rc_r = (n+r-1)c_{r-1};$$

$$\therefore \quad c_r = \frac{n+r-1}{r} c_{r-1} = \frac{n+r-1}{r} \cdot \frac{n+r-2}{r-1} c_{r-2} = \ldots$$

$$= \frac{n+r-1}{r} \cdot \frac{n+r-2}{r-1} \cdots \frac{n}{1} c_0.$$

But $$c_0 = 1;$$

$$\therefore \quad c_r = \frac{n(n+1)\ldots(n+r-1)}{r!}.$$

This demonstrates the identity of the two series in question.

Art. 3. The Logarithmic Series.

The series $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ is convergent if $x < 1$.

Hence by the Exponential Theorem

$$e^{\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)}$$
$$= 1 + \left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) + \frac{1}{2!}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^2 + \ldots$$

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Putting now $n=1$ on both sides of the identity of Art. 2, it follows that

$$1+\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)+\frac{1}{2!}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)^2+\ldots$$
$$=1+x+x^2+x^3+\ldots,$$

which is a Geometrical Progression to an infinite number of terms. Since $x<1$, its value is $\frac{1}{1-x}$;

$$\therefore\quad e^{\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)}=\frac{1}{1-x};$$

$$\therefore\quad \frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots=-\log_e(1-x);$$

$$\therefore\quad \log_e(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots \text{ if } x<1.$$

Art. 4. The Binomial Theorem.

Taking $x<1$,

$$(1-x)^{-n}=e^{-n\log_e(1-x)}$$
$$=e^{n\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)} \text{ when } x<1 \text{ by Art. 3}$$
$$=1+n\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$$
$$+\frac{n^2}{2!}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)^2+\ldots \text{ by Art. 1,}$$
$$=1+nx+\frac{n(n+1)}{2!}x^2+\ldots$$
$$+\frac{n(n+1)\ldots(n+r-1)}{r!}x^r+\ldots \text{ by Art. 2.}$$

Changing the sign of x, and putting $n=-m$,

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots$$
$$+\frac{m(m-1)\ldots(m-r+1)}{r!}x^r+\ldots, \text{ when } x<1,$$

which is the Binomial Theorem.

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(4) *On a method of finding the Potentials of Circular Discs by means of Bessel's Functions.* By A. B. BASSET, M.A.

1. THE present paper was commenced last summer, with the object of developing a method for finding the potential of an electrified circular disc by means of definite integrals involving Bessel's functions, when the disc is placed in a field of force whose potential is given. The same problem has been recently dealt with in a similar manner by Mr Gallop in the *Quarterly Journal**, but as my own method is somewhat more general than his, it may be worth while to lay this paper before the Society.

Let us take the normal through the centre of the disc as the axis of z, and employ cylindrical co-ordinates z, ρ and ϕ. The potential of the field of force may be expanded in a series of terms of the type

$$f_m(z, \rho) \cos m\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If V be the potential of the induced charge, V will consist of a series of which the term corresponding to (1) may be written

$$V_m = \Sigma_\lambda F(\lambda)\, \epsilon^{-\lambda z} J_m(\lambda\rho) \cos m\phi \ldots\ldots\ldots\ldots\ldots(2).$$

There is nothing to determine the value of λ excepting that it must be positive on the positive side of the disc, and we must therefore suppose λ to have all values from ∞ to 0, and replace the sum by a definite integral, whence

$$V_m = \cos m\phi \int_0^\infty \epsilon^{-\lambda z} F(\lambda)\, J_m(\lambda\rho)\, d\lambda \ldots\ldots\ldots\ldots\ldots(3).$$

At the surface of the disc $f_m(z, \rho)$ will be a given function of ρ only, say $-\phi(\rho)$, hence if c be the radius of the disc, we must have

$$\phi(\rho) = \int_0^\infty F(\lambda)\, J_m(\lambda\rho)\, d\lambda \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

when $\rho < c$.

The density which is proportional to $-dV/dz$ must vanish when $\rho > c$, hence we must have

$$0 = \int_0^\infty \lambda F(\lambda)\, J_m(\lambda\rho)\, d\lambda \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

when $\rho > c$; and the solution of the problem consists in determining $F(\lambda)$ so as to satisfy (4) and (5) when $\phi(\rho)$ is given.

I have only succeeded in obtaining the solution of this general problem in the two cases in which $m = 0$ or 1. The first case can be deduced by means of the theorem of Art. 2 and a theorem of Mr Gallop's; the second case is dealt with in Art. 5.

* Vol. XXI., p. 229.

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2. The first step is to establish a theorem analogous to Fourier's theorem, for expressing any function in the form of a definite integral involving Bessel's functions. The theorem is as follows:

If p and q be any positive real quantities, and $\phi(\rho)$ is a function which is finite and continuous for all values of ρ which lie between the limits p and q, but which is not necessarily finite at the limits, then the definite integral

$$\int_0^\infty d\lambda \int_q^p \lambda u \phi(u) J_m(\lambda u) J_m(\lambda \rho)\, du$$

is equal to $\phi(\rho)$ when ρ lies between the limits p and q, and is equal to zero when ρ lies beyond these limits *.

In order to prove the theorem, consider a thin plane conductor bounded by two concentric circles of radii p and q, which is electrified in such a manner that the density on either side is equal to

$$\tfrac{1}{2}\phi(\rho)\cos m\phi.$$

The potential will be

$$V = \int_q^p \int_\phi^{2\pi+\phi} \frac{u\phi(u)\cos m\phi' du d\phi'}{\{z^2+\rho^2+u^2-2\rho u\cos(\phi'-\phi)\}^{\frac{1}{2}}}.$$

Let $$\phi' - \phi = \eta$$
$$R^2 = \rho^2 + u^2 - 2\rho u\cos\eta.$$

Then $$V = \int_q^p \int_0^{2\pi} \frac{u\phi(u)(\cos m\phi\cos m\eta - \sin m\phi \sin m\eta)\, du d\eta}{(z^2+R^2)^{\frac{1}{2}}}.$$

The second integral vanishes, and the first is equal to

$$2\cos m\phi \int_0^\infty d\lambda \int_q^p du \int_0^\pi \epsilon^{-\lambda z} u\phi(u)\cos m\eta J_0(\lambda R)\, d\eta \dagger.$$

Now $$J_0(\lambda R) = J_0(\lambda\rho) J_0(\lambda u) + 2\Sigma_1^\infty J_m(\lambda\rho) J_m(\lambda u)\cos m\eta,$$

whence $$V = 2\pi\cos m\phi \int_0^\infty d\lambda \int_q^p \epsilon^{-\lambda z} u\phi(u) J_m(\lambda u) J_m(\lambda\rho)\, du.$$

The density $$= -\frac{1}{4\pi}\frac{dV}{dz},$$

hence this quantity must be equal to $\frac{1}{2}\phi(\rho)\cos m\phi$ when $p > \rho > q$, and must be zero when ρ lies beyond the limits p and q, whence

$$\int_0^\infty d\lambda \int_q^p \lambda u\phi(u) J_m(\lambda u) J_m(\lambda\rho)\, du = \phi(\rho) \quad p > \rho > q \quad \ldots\ldots\ldots\ldots (6)$$

$$= 0 \quad \left.\begin{matrix} \rho > p \\ \text{or } \rho < q \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6\text{A}).$$

* The particular case of $n=0$, $q=0$, $p=\infty$ is given in Heine, *Kugelfunctionen*, Vol. II., p. 299.

† Since $\int_0^\infty \epsilon^{-\lambda z} J_0(\lambda R)\, d\lambda = (z^2+R^2)^{-\frac{1}{2}}$.

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3. If a charged conductor of the form which we are considering be placed in a field of force, the density will usually be infinite at the edges, but dV/dz will always be finite except at the edges; whence although it is necessary that $\phi(\rho)$ should be finite and continuous between the limits p and q, it is not in general necessary that it should be finite at the limits. There are however two special cases, viz. (i) $q = 0$, p finite; and (ii) $p = \infty$, q finite, which require separate consideration.

The first case is that of a circular disc of radius p; and if $\phi(\rho)$ became infinite when $\rho = 0$, there would be a singular point at the origin.

The second case is that of an infinite plane screen having a circular aperture, and if $\phi(\rho)$ became infinite when $\rho = \infty$, the density would be infinite at an infinite distance from the aperture, which seems to be physically impossible.

If therefore in the first case $\phi(\rho) = \infty$ when $q = 0$; and in the second case $\phi(\rho) = \infty$ when $p = \infty$, the theorem could not be safely employed.

It must also be borne in mind that although the reduction of the integral in the form given may not always be easy to effect, yet as a matter of fact, the integral is really the limit of

$$\int_0^\infty \int_q^p \epsilon^{-\lambda z} \lambda u \phi(u) J_m(\lambda u) J_m(\lambda \rho)\, du,$$

when $z = 0$, and we may therefore reduce this latter integral to a simpler form, whenever it is possible to do so, and then put $z = 0$.

4. If $\phi(\rho)$ is finite and continuous for all values of ρ between 0 and ∞ inclusive, we may put $p = \infty$, $q = 0$, and the theorem becomes

$$\phi(\rho) = \int_0^\infty d\lambda \int_0^\infty \lambda u \phi(u) J_m(\lambda u) J_m(\lambda \rho)\, du$$

for all positive values of ρ. In order to solve the problem of finding the potential of an electrified circular disc which is placed in a field of force, we must determine a potential function $U_m \cos m\phi$ which satisfies the conditions that when $z = 0$,

$$U_m = J_m(\lambda \rho) \qquad \rho < c,$$

$$\frac{dU_m}{dz} = 0 \qquad \rho > c,$$

and the potential of the disc will be

$$\Sigma \cos m\phi \int_0^\infty d\lambda \int_0^\infty U_m \lambda u \phi(u) J_m(\lambda u) J_m(\lambda \rho)\, du \ldots.(7).$$

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The case of $m = 0$ has been solved by Mr Gallop, who finds that

$$U_0 = \frac{2}{\pi} \cdot \int_0^\infty d\mu \int_0^c \epsilon^{-\mu z} \cos \lambda v \cos \mu v J_0(\mu\rho)\, dv \ldots\ldots\ldots(8).$$

If therefore we put $m = 0$ in (7) and substitute the above value of U_0, we find that the integral

$$V_0 = \frac{2}{\pi} \cdot \int_0^\infty d\mu \int_0^\infty d\lambda \int_0^\infty du \int_0^c \epsilon^{-\mu z} \lambda u \phi(u) \cos \lambda v \cos \mu v J_0(\lambda u)\, J_0(\mu\rho)\, dv \ldots\ldots\ldots(9)$$

satisfies the conditions that

$$\left.\begin{aligned} V_0 &= \phi(\rho), & \rho < c, \quad z = 0 \\ \frac{dV_0}{dz} &= 0, & \rho > c, \quad z = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(10).$$

5. In order to find U_1, consider the integral

$$W_1 = \frac{2}{\pi} \cdot \int_0^\infty d\mu \int_0^c \epsilon^{-\mu z} \sin \lambda v \sin \mu v J_1(\mu\rho)\, dv.$$

Now

$$\int_0^\pi \int_0^\infty \epsilon^{-\mu\zeta} \cos(\mu\rho \cos\theta) \sin^2\theta\, d\theta\, d\mu = \int_0^\pi \frac{\zeta \sin^2\theta\, d\theta}{\zeta^2 + \rho^2 \cos^2\theta} = \frac{\pi}{\rho^2}\{(\zeta^2 + \rho^2)^{\frac{1}{2}} - \zeta\}.$$

Differentiating with respect to ζ, we obtain

$$\int_0^\infty \epsilon^{-\mu\zeta} J_1(\mu\rho)\, d\mu = \frac{1}{\rho}\left\{1 - \frac{\zeta}{(\zeta^2 + \rho^2)^{\frac{1}{2}}}\right\}.$$

Let $\zeta = z - \iota v$ where $\iota = \sqrt{-1}$, then

$$\iota \int_0^\infty \epsilon^{-\mu z} \sin \mu v J_1(\mu\rho)\, d\mu = \text{imaginary part of } -\frac{1}{\rho} \cdot \frac{z - \iota v}{\{(z - \iota v)^2 + \rho^2\}^{\frac{1}{2}}}.$$

Transform the left-hand side into polar co-ordinates, and put

$$R \cos 2\chi = r^2 - v^2,$$

$$R \sin 2\chi = 2rv \cos\theta,$$

so that

$$R = (r^4 + v^4 + 2r^2v^2 \cos 2\theta)^{\frac{1}{2}},$$

and we shall obtain

$$\int_0^\infty \epsilon^{-\mu z} \sin \mu v J_1(\mu\rho)\, d\mu = \frac{1}{R\rho\sqrt{2}}\{v(R + r^2 - v^2)^{\frac{1}{2}} - z(R - r^2 + v^2)^{\frac{1}{2}}\}.$$

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When $z = 0$ and $v > \rho$, then

$$R = v^2 - \rho^2,$$

and the integral vanishes.

When $z = 0$ and $\rho > v$, then

$$R = \rho^2 - v^2,$$

and the integral $$= \frac{v}{\rho\,(\rho^2 - v^2)^{\frac{1}{2}}}.$$

Hence when $\rho < c$, the limiting value when $z = 0$ of the integral,

$$W_1 = \frac{2}{\pi}\int_0^\rho \frac{v \sin \lambda v\, dv}{\rho\,(\rho^2 - v^2)^{\frac{1}{2}}}$$

$$= \frac{2\lambda}{\pi\rho} \cdot \int_0^\rho \sqrt{\rho^2 - v^2} \cos \lambda v\, dv$$

$$= J_1(\lambda\rho).$$

Again,

$$\iota \int_0^\infty \mu \epsilon^{-\mu z} \sin \mu v J_1(\mu\rho)\, d\mu = \text{imaginary part of } \frac{\rho}{\{(z - \iota v)^2 + \rho^2\}^{\frac{1}{2}}}$$

$$= \frac{\iota\rho}{R^{\frac{1}{2}}} \sin 3\chi$$

$$= 0 \text{ when } z = 0 \text{ and } \rho > v,$$

$$\therefore \text{ when } z = 0,\ \rho > c,$$

$$\frac{dW_1}{dz} = 0,$$

whence $$U_1 = W_1.$$

Hence, if

$$V_1 = \frac{2}{\pi} \cdot \int_0^\infty d\mu \int_0^\infty d\lambda \int_0^\infty du$$

$$\int_0^c \epsilon^{-\mu z} \lambda u\, \phi(u) \sin \lambda v \sin \mu v J_1(\lambda u)\, J_1(\mu\rho)\, dv \quad \ldots\ldots\ldots (11),$$

then

$$\left.\begin{aligned} V_1 &= \phi(\rho), \quad \rho < c, \quad z = 0 \\ \frac{dV_1}{dz} &= 0, \qquad \rho > c, \quad z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12).$$

6. If the conductor consists of an infinite plane screen having a circular aperture of radius c, the solution can be obtained when the potential is symmetrical with respect to the axis of the circular aperture. In this case the problem may be stated as follows.

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If $\phi(\rho)$ be any given function of ρ which is finite and continuous for all values of ρ between the limits infinity and c, it is required to find a function of $F(\lambda)$, such that

$$\left.\begin{aligned} \phi(\rho) &= \int_0^\infty F(\lambda) J_m(\lambda\rho)\, d\lambda, \quad \rho > c \\ 0 &= \int_0^\infty \lambda F(\lambda) J_m(\lambda\rho)\, d\lambda, \quad \rho < c \end{aligned}\right\} \ldots\ldots (13).$$

7. In the case of $m = 0$, consider the integral,

$$W = \frac{2}{\pi}\int_0^\infty d\mu \int_c^\infty \epsilon^{-\mu z} \sin \lambda v \sin \mu v J_0(\mu\rho)\, dv.$$

By a known formula

$$\int_0^\infty \epsilon^{-\mu\zeta} J_0(\mu\rho)\, d\mu = \frac{1}{(\zeta^2+\rho^2)^{\frac{1}{2}}}.$$

Let $\zeta = z - \iota v$, where $\iota = \sqrt{-1}$, then

$$\iota\int_0^\infty \epsilon^{-\mu z} \sin \mu v J_0(\mu\rho)\, d\mu = \text{imaginary part of } \frac{1}{\{(z-\iota v)^2+\rho^2\}^{\frac{1}{2}}},$$

and if we transform to polar co-ordinates, and put

$$R\cos 2\chi = r^2 - v^2,$$
$$R\sin 2\chi = 2rv\cos\theta,$$

as in Art. (5), we obtain

$$\int_0^\infty \epsilon^{-\mu z} \sin \mu v J_0(\mu\rho)\, d\mu = \frac{(R^2+v^2-r^2)^{\frac{1}{2}}}{R\sqrt{2}}$$

when $z = 0$ and $\rho > v$ the integral vanishes; but when $z = 0$ and $\rho < v$, the integral $= \dfrac{1}{\sqrt{v^2-\rho^2}}$.

Hence when $\rho > c$, the limiting value of the integral W is

$$\frac{2}{\pi}\cdot\int_\rho^\infty \frac{\sin\lambda v dv}{\sqrt{v^2-\rho^2}}$$
$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda\rho\cosh\theta)\, d\theta$$
$$= J_0(\lambda\rho)$$

by a known formula.

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Also

$$\epsilon \int_0^\infty \mu\epsilon^{-\mu z} \sin \mu v J_0(\mu\rho)\, d\mu = \text{imaginary part of } \frac{z - iv}{\{(z-iv)^2 + \rho^2\}^{\frac{1}{2}}}$$

$$= \frac{c}{R^{\frac{1}{2}}} (z \sin 3\chi - v \cos 3\chi)$$

$$= 0 \text{ when } z = 0 \text{ and } \rho < v,$$

$$\therefore \text{ when } z = 0 \text{ and } \rho < c$$

$$\frac{dW}{dz} = 0.$$

Hence, if $V = \frac{2}{\pi} \int_0^\infty d\lambda \int_0^\infty du \int_0^\infty dv$

$$\int_0^\infty \epsilon^{-\mu z} \lambda u \phi(u) \sin \lambda v \sin \mu v J_0(\lambda u) J_0(\mu\rho)\, d\mu \quad \ldots\ldots (14),$$

then

$$\left.\begin{aligned} V &= \phi(\rho) \text{ when } z = 0 \text{ and } \rho > c \\ \frac{dV}{dz} &= 0 \text{ when } \quad z = 0 \text{ and } \rho < c \end{aligned}\right\} \ldots\ldots\ldots (15).$$

8. In the last article we have quoted the known formula

$$J_0(\lambda) = \frac{2}{\pi} \int_0^\infty \sin(\lambda \cosh\theta)\, d\theta.$$

This result may be easily established by comparing the results obtained by integrating the definite integral

$$\int_0^\infty du \int_0^\infty \cos x \cos u^2 (x^2 - \lambda^2)\, dx$$

with regard to x and u respectively. If we first perform the integration with respect to u, the result

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} \int_\lambda^\infty \frac{\cos x\, dx}{(x^2-\lambda^2)^{\frac{1}{2}}} + \frac{1}{2}\sqrt{\frac{\pi}{2}} \int_0^\lambda \frac{\cos x\, dx}{(\lambda^2 - x^2)^{\frac{1}{2}}} \ldots\ldots (16).$$

The second integral on the right-hand side is equal to $\pi J_0(\lambda)/2$; and if in the first integral we put $x = \lambda \cosh\phi$, and afterwards make ϵ^ϕ successively equal to $2\lambda u^2$ and $1/2\lambda u^2$ and add the results, we shall find that

$$\int_\lambda^\infty \frac{\cos x\, dx}{(x^2-\lambda^2)^{\frac{1}{2}}} = \int_0^\infty \cos\left(\lambda^2 u^2 + \frac{1}{4u^2}\right) \frac{du}{u} \ldots\ldots (17).$$

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In the same way we can show that

$$\int_\lambda^\infty \frac{\sin x\,dx}{(x^2-\lambda^2)^{\frac{1}{2}}} = \int_0^\infty \sin(\lambda\cosh\phi)\,d\phi$$

$$= \int_0^\infty \sin\left(\lambda^2u^2+\frac{1}{4u^2}\right)\frac{du}{u} \quad \text{.....(18).}$$

Whence, the right-hand side of (16),

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^\infty \cos\left(\lambda^2u^2+\frac{1}{4u^2}\right)\frac{du}{u} + \frac{\pi^{\frac{1}{2}}J_0(\lambda)}{4\sqrt{2}} \quad \text{.....(19).}$$

Now if we integrate the original definite integral, with respect to x, the result

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^\infty \cos\left(\lambda^2u^2+\frac{1}{4u^2}\right)\frac{du}{u} + \frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^\infty \sin\left(\lambda^2u^2+\frac{1}{4u^2}\right)\frac{du}{u},$$

whence by (18) and (19),

$$J_0(\lambda) = \frac{2}{\pi}\int_0^\infty \sin\left(\lambda^2u^2+\frac{1}{4u^2}\right)\frac{du}{u}$$

$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda\cosh\phi)\,d\phi.$$

9. As an example of the formulæ of Art. (4), consider the case of an uninfluenced disc which is electrified to unit potential: then

$$V = \frac{2}{\pi}\int_0^\infty d\lambda\int_0^\infty du\int_0^\infty dv\int_0^\infty \epsilon^{-\mu z}\,u\lambda\cos\lambda v\cos\mu v J_0(\lambda u)\,J_0(\mu v)\,d\mu.$$

Now $$\int_0^\infty u\epsilon^{-\beta u}J_0(\lambda u)\,du = \frac{\beta}{(\lambda^2+\beta^2)^{\frac{1}{2}}}.$$

Therefore

$$\int_0^\infty du\int_0^\infty u\lambda\cos\lambda v J_0(\lambda u)\,d\lambda = \text{lim of } \left[\int_0^\infty \frac{\lambda\beta\cos\lambda v\,d\lambda}{(\beta^2+\lambda^2)^{\frac{3}{2}}}\right]_{\beta=0}.$$

Put $\lambda = \beta y$, and the integral

$$= \text{lim of } \left[\int_0^\infty \frac{\cos\beta yv\,dy}{(1+y^2)^{\frac{3}{2}}}\right]_{\beta=0}$$

$$= 1, \text{ when } \beta = 0.$$

Hence $$V = \frac{2}{\pi}\int_0^\infty \frac{\epsilon^{-\mu z}}{\mu}\sin\mu c J_0(\mu\rho)\,d\mu,$$

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which is Weber's result*. This integral can be easily evaluated in the form

$$V = \frac{2}{\pi} \sin^{-1} \frac{2c}{p+q},$$

when

$$p^2 = r^2 + c^2 + 2rc \sin\theta,$$

$$q^2 = r^2 + c^2 - 2rc \sin\theta,$$

which is the well-known expression for V.

15th April, 1886.

May 24, 1886.

PROFESSOR FOSTER, PRESIDENT, IN THE CHAIR.

Mr H. F. NEWALL, M.A., Trinity College, was duly elected a Fellow of the Society, and Mr PERCY GROOM, of Trinity College, an Associate.

The following communications were made to the Society:—

(1) *On a New Species of Dinophilus.* By W. F. R. WELDON, M.A.

(2) *On the life-history of Pedicellina.* By S. F. HARMER, B.A.

(3) *On the organ of attachment of Laminaria bulbosa.* By WALTER GARDINER, M.A.

The author stated that correlated with the flattened stem and enormous leaf of this plant there were especial mechanical arrangements to enable the organism to withstand the force of the wave-beats to which it was continually exposed. The additional stability was chiefly attained in two ways. First, the flattened stem twisted upon itself at its basal portion; and secondly, there was developed just above the root a meristematic ring of tissue which rapidly grew out on all sides, until finally in its adult state it formed the well-known irregular bulbous structure so characteristic of the species. The true root persists for some time, but gradually atrophies and disappears. From the bulbous portion, adventitious roots arise which function in every way as root organs, and hold the plant firmly to the substratum.

* Borchardt's *Journal*, Vol. LXXV., p. 77. See also Heine, Vol. II., p. 192.

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